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PROGRESSIVE LESSONS

IN

APPLIED SCIENCE.

PART III.—TRIGONOMETRY, VISION, SURVEYING INSTRUMENTS.

BY

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LONDON:

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NEW YORK:

446, BROOME STREET.

1875.

NEW YORK
PUBLIC
LIBRARY

INTRODUCTION.

LOOKING back over what has been done in the two previous parts we see that an attentive examination of very simple matters has enabled us to answer difficult and useful questions. Knowing the principal geometric law, that the three angles of a trigon make together half a turn, we have been able, by the help also of a little knowledge of arithmetic, to calculate the surfaces, bulks, and weights of various bodies; and knowing the peculiar way in which three pressures resist each other, we have learned how to compute beforehand the strains in various structures. It is hence clear that the builder or the mechanician can hardly prosecute his business without knowing something of Geometry, while the geometer himself can make scarcely any progress without numbers. A more thorough knowledge of geometry and more skill in arithmetical computation must then greatly increase our knowledge of business matters.

When there are linear dimensions given in sufficient number to define a figure we are able to calculate the surface; but we have not yet learned how to make the analogous calculations when angles form part of the data, except in the single case of the right angle; nor are we able *to tell the angles* when the sides are known, other-

wise than by constructing the figure and using the protractor. We therefore now proceed to study the computation of angles. This branch of geometry is commonly known under the title *Trigonometry*, but would be better described by leaving off the first syllable of the word, so as to give it the meaning "measurement of angles." Our real object is to connect the measurement of lines with the measurement of angles.

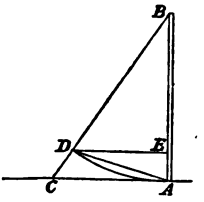
All magnitudes must be measured by magnitudes of their own kinds, yet they are often measured by means of magnitudes of another kind. Thus in order to measure angles we assume the whole revolution as the unit; divide it into some number of equal parts and examine how many of those go to make up the angle which is to be measured. But to perform this operation conveniently, we use a circle having its circumference divided into equal parts; so that we measure angles by means of the graduated circumference. The arc furnishes us with a convenient means for measuring an angle; we often say "The arc measures the angle," but this is only an ellipsis or shortening of words; an arc is a line, and angles must be measured by angles; we should rather say "the arc serves to measure the angle."

Now in measuring angles we may use protractors of different sizes; so that the same angle may be measured by means of arcs of very different lengths, these lengths being proportional to the radii of the protractors; and thus we see that it is not so much the arc, as the ratio which the arc bears to the whole circumference, to the radius, or to (say) the diameter of the protractor, which serves to measure the angle. Thus the length of an arc may be stated in two ways; it may be referred to the whole circumference, or it may *be compared* with the radius; and as we shall have

to use both of these comparisons, we must be able to translate the one into the other.

If, as is usual and convenient, we take the radius of the circle as unit, the semicircumference is 3·141 5926, and therefore the length of an arc of one ancient degree is the 180th part of this or ·017 4533, and that of one minute ·000 2909. Or if we use the modern division of the circumference into 400 grades, one grade is ·015 7080 and one minute (the hundredth part of a grade) is ·000 1571. Thus it is an easy matter to make a table of the lengths of circular arcs.

In order to observe the angular altitude of the sun or of a star, the ancient astronomers used a style or *gnomon* A B set up in the middle of a flat arena; they noted the length of the shadow A C of this gnomon, and, knowing the height, were able to get the angle A B C made by an upright line and the line drawn to the sun. This they could do by making, on a board, a trigon similar to A B C and measuring the angle by means of a protractor. It would be much better, however, to compute the length of the arc A D from the length of the tangent A C, if the mode of making the calculation could be found out.



We already know how to calculate the length of C B, whence the horizontal line D E, the distance A E, and the chord A D can be found.

By making a table of the lengths of the chords of arcs, the ancient trigonometers could find out the arc which has a given chord, and so get the angle A B C called the *zenith distance* of the sun or star. The Arabian astronomers made a great improvement in the mode of calcula-

tion, by using a table of the horizontal lines D E corresponding to different arcs, as also one of the tangents A C. To A C they gave the name ظلّ *zil* or shadow, and to D E جيب *jeïb* or purse (worn in the bosom), we name them *tangent* and *sine* (Latin *sinus* the bosom). At first, trigonometers confined their attention to arcs less than a quadrant.

In modern times we have somewhat improved and greatly extended these tables, but the mode of proceeding in trigonometrical calculations remains the same; it consists of, first, the construction of these tables and afterwards of their use. Now the using of the tables is a much more easy matter than the construction thereof; nay more, we can hardly understand the mode of construction until we know something of the use, so that we are forced to proceed in a somewhat unnatural order, and to use tables before we well know how to compute them.

PROGRESSIVE LESSONS

IN

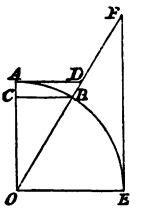
APPLIED SCIENCE.

PART III.—TRIGONOMETRY, VISION, SURVEYING INSTRUMENTS.

LESSON I.

ON THE TRIGONOMETRICAL LINES.

O A BEING the radius of a circle and A B a portion of its circumference, the perpendicular B C drawn from B to the radius O A is called the *sine* of the arc A B; the tangent applied at A and continued to meet the extension of the radius O B in D is called the *tangent* of the arc A B; and the line O D intercepted by the tangent is called the *secant* of A B. As these terms are of very frequent use, we abbreviate them into *sin, tan, sec;* thus we write C B = sin A B; A D = tan A B; O D = sec A B.



The defect of an arc from the whole quadrant is called its *complement,* thus B E is the complement of A B, and, of course, A B is the complement of B E. The sine, the tangent and the secant of B E are called the cosine, the

cotangent and the *cosecant* of A B, these words being abbreviations, thus cosine being a shortening of "sine of the complement." In writing, we still further shorten them into *cos*, *cot*, *cosec* or better *cse*, thus O C, which is equal to the perpendicular drawn from B to O E, is the sine of B E or the cosine of A B; E F is the tangent of B E or the cotangent of A B; and O F is the secant of B E or the cosecant of A B, and we write shortly O C = cos A B; E F = cot A B; O F = cse A B. Conversely we may write C B = cos B E; A D = cot B E; O D = cse B E; O C = sin B E; E F = tan B E; O F = sec B E.

As self-reliance is essential to all progress in knowledge, it is proper that each student should for himself construct a small table of sines, tangents and secants. For this purpose he may describe a quadrant with a radius O A equal to one hundred divisions of some convenient scale, so that he may be able to read to thousandth parts of the radius. He may next divide the quadrant carefully into equal parts, say each to be five degrees of the ancient or of the modern division, and draw the sines of the arcs. Measuring these carefully, he may write their values in a previously prepared page; and thereafter, but not until he have written them, he may compare them with the values given in any printed table.

He might proceed in the same way to make tables of the tangents and secants; but it is better to compute these from the values of the sines. For this purpose we must examine the relations of these trigonometric lines to each other.

In the first place we observe that the squares of O C and of C B make together the square of the radius, or that

$$\sin AB^2 + \cos AB^2 = R^2,$$

wherefore if one of these be found by measurement, the other may be got by calculation; in this way we might have spared the measurement of the sines of one half of the quadrant.

Next we may observe that the trigon O A D is similar to O C B and that therefore

$$\cos AB : \sin AB :: R : \tan AB;$$

now in all trigonometrical tables the radius is assumed as the unit of comparison, wherefore the numerical representative of the tangent is the quotient obtained by dividing that of the sine by that of the cosine; or, as we write it concisely,

$$\tan AB = \frac{\sin AB}{\cos AB},$$

and

$$\cot AB = \frac{\cos AB}{\sin AB}.$$

By simple division the student may now complete his table of tangents; this he should do from his own measurements of the sines, in order to study the effects of slight errors.

And, in order to construct the table of secants, we notice that O C : O B :: O A : O D or

$$\cos AB : R :: R : \sec AB$$

and also $\sin AB : R :: R : \operatorname{cse} AB$, which may be written $\sec AB = \frac{1}{\cos AB}$, $\operatorname{cse} AB = \frac{1}{\sin AB}$, and enable us to make a table of secants from that of sines.

When any one of the six lines connected with an arc and its complement is given, the remaining five may be computed. It is in general convenient to express these lines in decimal parts of the radius, but in some cases we

may find it more suitable to use other fractions. Thus if the height of a gnomon had been 589 inches and the length of the shadow 300, we should have found the angle to be almost exactly 27° or 30ᵍ, wherefore, the arc A B being three tenth parts of the quadrant, its tangent A D must be very nearly $\frac{300}{589}$ parts of the radius; or putting a simply for the arc, $\tan a = \frac{300}{589}$ when the radius is unit. But $OD^2 = OA^2 + AD^2$ or

$$\sec a^2 = 1 + \tan a^2$$

so that we may obtain the value of the secant, from that of the tangent. Making the calculation in this case we get $\sec a = \frac{661}{589}$. Also observing that DA:AO :: OE : EF or that

$$\tan a : R :: R : \cot a$$

we find here $\cot a = \frac{589}{300}$. The student should, by this time, find no difficulty in computing the sine and cosine. He may convert the common fractions into decimals and compare the results with his printed table; they will be found to agree very nearly with the numbers given for 27°.

The arc A B serves to measure the angle A O B, and we often call B C the sine of the angle A O B, A D its tangent, and so on; but this is a figure of speech, what we really mean to say is that B C is the sine of the arc which serves to measure the angle A O B. This arc is not *necessarily* described with a radius *unit*, though it be convenient to take unit for the radius.

Exercise 1.

Construct and measure the angle whose tangent is $\frac{36}{77}$ of the radius; compute its secant, sine, cosine, cotangent and cosecant; convert the fractions into decimals and compare with any printed table.

Exercise 2.

Construct and measure the angle whose sine is $\frac{20}{29}$; compute the other *functions*, viz. the cosine, tangent, cotangent, secant and cosecant.

Exercise 3.

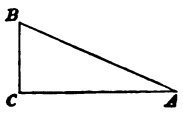
The secant of an angle is $\frac{65}{63}$, compute thence its tangent and its sine; construct the angle itself from each of these in order to compare the relative degrees of precision.

By help of theodolites fitted with telescopic sights we are able easily to measure angles in the field to within one minute of a degree, nay even in the case of geodetical circles, to within the sixth part of a minute; but on paper we may easily err by several minutes. Hence the graphic method is quite useless for the formation of trigonometric tables for business purposes, and we must have recourse to arithmetical operations; we shall however defer the consideration of these until we shall have made some more progress in the art of calculation.

LESSON II.

ON THE RIGHT-ANGLED TRIGON.

A B C BEING a trigon right angled at C, if with the radius A B from the centre A we describe a circle, the arc intercepted between the lines A B, A C serves to measure the angle at A, and B C is the sine of that arc, A C its cosine, and therefore we have the two proportions



$$A B : B C :: R : \sin A$$
$$A B : A C :: R : \cos A$$

where R is put for the radius of the tables (usually unit) and A for the arc which serves to measure the angle A.

If, however, we describe the circle with the radius A C, C B becomes the tangent of the intercepted arc, and A B becomes its secant, wherefore we get the two proportions

$$A C : C B :: R : \tan A$$
$$A C : A B :: R : \sec A.$$

And again if the centre be placed at B, and the radius B C be used, C A is the tangent and B A the secant of the arc which measures the angle A B C, now this arc is the complement of that which measures C A B, wherefore

$$B C : C A :: R : \cot A$$
$$B C : B A :: R : \operatorname{cse} A.$$

In order that the student may clearly see the meaning of these six proportions we shall suppose the hypotenuse A B to be 89, and the altitude B C to be 39; in which case A C is necessarily 80. When A B = 89 is made the radius B C = 39 is the sine of the angle A (as we say for shortness); but the tabular radius is only unit, where-

fore the tabular sine must be the 89th part of 39 or $\frac{39}{89} = \cdot 438\ 2022$; at the same time C A = 80 becomes the cosine, wherefore the tabular cosine is $\frac{80}{89} = \cdot 898\ 8764$. When A C is made the radius C B becomes the tangent and A B the secant so that tan $A = \frac{39}{80} = \cdot 487\ 5000$, sec $A = \frac{89}{80} = 1 \cdot 112\ 5000$; and when B C is taken as the radius C A is the tangent of B or the cotangent of A, while B A is the cosecant of A, wherefore cot $A = \frac{80}{39} = 2 \cdot 051\ 2820$; cse $A = \frac{89}{39} = 2 \cdot 282\ 0512$. On inspecting a table it will be found that these six values are stated as belonging to an arc almost exactly of 26°, wherefore we conclude that the angle A in this case has that value.

Thus we see that when two dimensions, beside the right angle, are given, we may compute the rest by aid of a table of circular parts. We shall take the cases in order.

Case 1.

When the hypotenuse and an angle are given.

Let A be the given angle then

R : sin A :: A B : B C; R : cos A :: A B : A C, or, when the tabular radius is unit,

$$B C = A B . \sin A ; A C = A B . \cos A.$$

For example let the hypotenuse be 7373 and the angle 37° 28′. In the trigonometrical tables we find sin 37° 28′ = ·608 2998; cos 37° 28′ = ·793 7074; and multiplying each of these by 7373, we find 4484·993 ′

the side opposite to, and 5852·005 for the side adjacent to the given angle.

EXERCISE 1.

The hypotenuse being 4789 and one of the angles 15° 16′, required the two sides of the right angle.

EXERCISE 2.

The hypotenuse is 1289 and one of the angles 64° 15′, required the sides.

EXERCISE 3.

An angle being 42° 15′ and the hypotenuse 2225, what are the sides of the right angle?

CASE 2.

When a side and one of the angles are given.

Let A be the given angle and A C the given side, then

$$R : \tan A :: AC : CB;\quad R : \sec A :: AC : AB \text{ or}$$
$$CB = AC.\tan A;\quad AB = AC.\sec A.$$

Thus if the base be 989 and the adjacent angle 33° 43′, we look in the table for the tangent and secant of this angle, and find tan 33° 43′ = ·667 3374; sec 33° 43′ = 1·202 2226, which, multiplied by 989, give 659·997 for the altitude and 1188·998 for the hypotenuse.

When a side and the opposite angle are given we observe that the tangent of the adjacent is just the cotangent of the opposite angle; so that this need hardly be called another case.

EXERCISE 1.

One side being 156 and the angle adjacent to it 40° 27′, required the hypotenuse and the other side.

EXERCISE 2.

The base being 3132 and the adjacent angle 33° 31½′, required the altitude and the hypotenuse.

Exercise 3.

The base being 464 and the opposite angle 59° 9′ 20″, required the hypotenuse and the altitude.

Case 3.

When the hypotenuse and a side are given.

Since $AB : BC :: R : \sin A$, we have $\sin A = \frac{BC}{AB}$, for $R = 1$, and having obtained the tabular sine of A, we may find A itself by inspection of the table. Also A being thus found we may compute the remaining side by the formula $AC = AB . \cos A$. The side AC might have been computed directly by the method of squares.

For example if the hypotenuse be 7309 and the side BC be 5460, we obtain

$$\sin A = \frac{5460}{7309} = \cdot 747\ 0242$$

and on turning to the table we find this to be the sine of 48° 20′ almost exactly. The cosine of this is ·664 7959, which multiplied by 7309 gives 4858·993, almost 4859, for the length of AC. The direct calculation gives 4859 exactly; the minute error has arisen from this that the sine of 48° 20′ is ·747 0251, so that the angle A is, in reality, a little less than 48° 20′.

Exercise 1.

The hypotenuse being 4133 and the base 3555, required the angles and the altitude.

Exercise 2.

The hypotenuse is 4225 and the altitude 2016, what are the angles, and what the base?

Exercise 3.

The altitude being 1353 and the hypotenuse 7625, required the angles and the base.

Case 4.

When the two sides of the right angle are given.

From the proportion A C : C B :: R : tan A we obtain $\tan A = \frac{C B}{A C}$, whence A may be found and thereafter A B, since A B = A C. sec A.

As an example, we may suppose the two sides to be A C = 6960, C B = 5969, whence $\tan A = \frac{5969}{6960}$ = ·857 6149, and seeking for this among tangents we find ·857 6084 to be the tangent of 40° 37′. Without, for the present, paying any attention to the small difference between this and the computed tangent, we take out the secant of 40° 37′ which is 1·317 3808 and multiply it by 6960; the result 9168·97 is the length of the hypotenuse. On taking the square root of the sum of the squares of A C and C B we get 9169 exactly.

Exercise 1.

The two sides being 2924 and 693, required the angles and the hypotenuse.

Exercise 2.

The sides of the right angle being 2655 and 728, required the angles and the hypotenuse.

Exercise 3.

The sides are 493 and 276, what are the angles and what the hypotenuse?

LESSON III.

ON INTERPOLATION.

In most of the examples above given, the sines, tangents, secants of the angles have been found in the tables either exactly or very nearly so. This rarely happens in the course of business, and we have now to see what must be done when there is a considerable difference, or when the angle is given to less than one minute of a degree. On the large astronomical instruments the sixtieth part of the minute is counted.

For example if the hypotenuse be 3805 and one of the angles 39° 48′ 27″ we should have to get the sine and the cosine of the arc corresponding to this angle. On referring to any seven-place table of sines we find the sine of 39° 48′ to be ·640 1097 while that of 39° 49′ is ·640 3332; the sine of which we are in search must be between these two. Now the difference, that is the change for a whole minute or 60″, is 2235 and therefore we may make the proportion 60″: 27″:: 2235 : 1006, which gives 1006 as the change for 27″; and this added to the sine of 39° 48′ makes ·640 2103 for the sine of 38° 48′ 27″.

Similarly the cosines of the two arcs are found to be ·768 2835 and ·768 0973, giving 1862 for the change corresponding to one minute, whence, by the proportion 60″: 27″:: 1862 : 838, we find the change for 27″ to be 838. Here we must be careful to notice that the cosine decreases while the arc increases, and that this 838 must be deducted from, not added to, the cosine of 39° 48′. Hence the cosine of 39° 48′ 27″ is ·768 1997. On multiplying these by 3805 we obtain 2436 for the side opposite to the given angle and 2923 for the side adjacent to it.

Here we have been taking for granted that the change in the sine is proportional to the change in the arc. But this is not strictly true; it happens in this case that the deviation from proportionality is so small as to be practically unnoticeable. In most of the tables the differences of the sines are given; on glancing at these we see that they slowly lessen as the arc grows greater, thus the next difference in the sines is 2234 being 1 less, and the next in the cosines is 1863 or 1 more. From end to end of the table of sines no appreciable error arises from *interpolating* (as it is called) by simple proportion; which is to say that the arc of one minute has no perceptible curvature, that is, perceptible in the seventh decimal place.

Again let us suppose the hypotenuse to be 5765 and one of the sides 2516. In this case the sine of the opposite angle is $\frac{2516}{5765} = \cdot 436\,4267$. The sine immediately less than this is $\cdot 436\,2784$ which belongs to 25° 52′. The difference between this and the next sine is 2617, while the difference between it and the given sine is 1483. Wherefore in order to get the number of seconds we have the proportion 2617 : 1483 :: 60″ : 34″ so that the angle must be 25° 52′ 34″. From this the student may compute the remaining side.

When the differences change quickly the interpolation becomes troublesome. We may take as an illustration the angle 84° 26′ 30″ and the adjacent side 515; in which case we have to find the tangent and the secant.

By inspecting the table we get

tan 84° 26′	= 10·260 249	31 006	186
27	= 10·291 255	31 192	188
28	= 10·322 447	31 380	
29	= 10·353 827		

from which it is seen that the differences change very rapidly, the second difference being 186. In such a case it will not do to make a simple proportion. Here the second differences hardly change; we may suppose them to be constant. Now we have already had an example of constant second differences in the table of square numbers, and so may expect some help from that quarter.

The beginning of a table of square numbers with their first and second differences is

Num.	Sq.	1st Dif.	2nd Dif.
0	0	1	2
1	1	3	2
2	4	5	
3	9		

but our second difference is 186, wherefore if we were to construct a table of 93 times each square number we should obtain the same second difference, as in this way

Num.	$93 \times n^2$.	1st Dif.	2nd Dif.
0	0	93	186
1	93	279	186
2	372	465	
3	837		

If we subtract these from the corresponding numbers in the preceding scheme of tangents we leave, counting from 26′,

0	10·260 249	30913	0
1	10·291 162	30913	2
2	10·322 075	30915	
3	10·352 990		

where the second difference almost is gone, and where the interpolation may be by simple proportion. If we put n for the number of minutes counting from 84° 26′, the tangent is thus given by the formula

$$10\cdot260\,249 + [illegible]\,n + 93\,n^2$$

In our present example n is 30″ or $\frac{30}{60}$ of one minute, so that the tangent of 84° 26′ 30″ is

$$10{\cdot}260\,249 + \frac{30}{60} \times 30\,913 + \left(\frac{30}{60}\right)^2 \times 93$$
$$= 10{\cdot}260\,249 + 15\,456{\cdot}5 + 23{\cdot}25$$
$$= 10{\cdot}275\,729$$

The student may apply the same method to the secant of the arc, and compute the sides of the trigon.

When the second differences themselves change perceptibly, we have to take the third differences into account, and this we can do by extending the same method. Thus if the side of a right-angled trigon be 83 and its adjacent angle 88° 37′ 10″, we have, for the purpose of computing the other sides, to get the tangent and the secant from the tables. Extracting several of the tangents and taking their successive differences we have

Arc.	Tangent.	1st Dif.	2nd Dif.	3rd Dif.	4th Dif.
88° 37′	41·410 588	505 202	12 472	467	26
38	41·915 790	517 674	12 939	493	
39	42·433 464	530 613	13 432		
40	42·964 077	544 045			
41	43·508 122				

where there are differences even of the fourth order.

For the moment we shall put aside the fourth difference and, to avoid fractions (as we are at present only seeking for the principle), we shall write 468 for the first of the third differences.

Now we had third differences when constructing a

table of cube numbers, and beginning at 0, the differences are

0	0	1	6	6
1	1	7	12	
2	8	19		
3	27			

but the third difference in the preceding scheme is 468 or 78 times that of the cubes, wherefore multiplying the cubes and their differences by 78 we get the table

0	0	78	468	468
1	78	546	936	
2	624	1482		
3	2106			

and subtracting these from the corresponding numbers in the table of tangents, there remain in the first line

88° 37′ 41·410 588 505 124 12 004

which lead to an interpolation by second differences. Subtracting from these 6 002 times the differences for squares, namely

0 6 002 12 004

there remain for interpolation by first differences

88° 37′ 41·410 588 499 122

wherefore if n be the number of minutes counted from 88° 37′, the tangent is given by the formula

$$\begin{aligned} \tan 88° 37' + n' = {} & 41·410\,588 \\ & + \cdot 499\,122\, n \\ & + 6\,002\, n^2 \\ & + 78\, n^3 \end{aligned}$$

In our example $n = 10''$ or $\frac{10}{60}$ wherefore

$$\begin{aligned} \tan 88° 37' 10'' = {} & 41·410\,588 \\ & + 83\,187 \\ & + 166·72 \\ & + \cdot 39 \\ = {} & 41·493\,942 \end{aligned}$$

The computation of the secant, and thereafter of the sides of the trigon, may be completed by the student.

The same principle may be extended to fourth and further differences, but as we shall afterwards have to explain more rapid processes, it is unnecessary to proceed farther at present.

If we have to find out the arc from its tangent or from its secant, the computation becomes troublesome when second or third differences have to be taken into account. Thus if we wish to know the angles of a right-angled trigon whose sides are 861 and 7540, we have for the tangent of the larger angle the value $\frac{7540}{861} = 8{\cdot}757\ 2590$. The tangent of 83° 29″ is 8·754 2461 being deficient by 3 0129.

Taking out the successive tangents and their differences we have

83° 29′	8·754 2461	22 6413	1159
30	8·776 8874	22 7572	1168
31	8·799 6446	22 8740	
32	8·822 5186		

Reading the first of the second differences as 1160, the addition to the tangent of 83° 29′ takes the form

$$22\ 5833\ n + 580\ n^2$$

and we have to find out what n must be in order that the value of this may be 3 0129.

This can only be done by trial. When the desired equality is put in the form

$$3\ 0129 = 22\ 5833\ n + 580\ n^2$$

it is called an *equation*, and because the second power of *the unknown* quantity is found in it, it is said to be of the

second degree; or shortly, it is called a *quadratic equation.*

When we have to take account of differences of the third order we have to deal with an equation of the same degree, or as it is called a *cubic equation;* and thus it is seen that we must be acquainted with the whole doctrine of equations in order to be able to use *tables* completely.

In all cases, the use of second and third differences is troublesome; in order to avoid it we seek to make the tables so full as to render these higher differences very small; or we look for some other way of resolving the problem in hand.

In the present case, we may take the latter plan and instead of determining the larger angle by its tangent or by its secant we may determine it by the cotangent or by the sine.

Sometimes however there is no help. Thus the seaman has to find the error of his time-keeper by means of Lunar Distances, that is by the distances of the moon from the stars. These Lunar Distances are set down in the almanac for every third hour; and in order to find the distance at any intermediate time or the time corresponding to any intermediate distance, the navigator has to interpolate between these. The moon's motion is not quite equable and second differences have sometimes to be taken into account.

EXERCISE 1.

The hypotenuse is 1865 and the angle 5° 37′ 52″, required the base and the altitude.

EXERCISE 2.

The base being 609 and the adjacent angle 18° 10′ 50″, *required the altitude* and the hypotenuse.

EXERCISE 3.

The two sides are 208 and 105, required the angles and the hypotenuses.

EXERCISE 4.

The angle being 29° 47′ 13″ and the hypotenuse 9461, required the base and the altitude.

EXERCISE 5.

The base being 3225 and the adjacent angle 30° 20′ 45″, required the altitude and the hypotenuse.

EXERCISE 6.

The base being 6667 and the altitude 4644, required the angles and the hypotenuse.

EXERCISE 7.

The hypotenuse is 8237 and the altitude 4988, what are the angles and what the base?

EXERCISE 8.

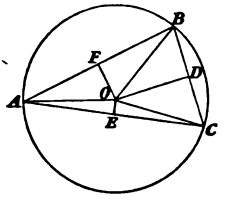
The angle being 38° 48′ 43″ and the opposite side 5456, required the adjacent side and the hypotenuse.

LESSON IV.

ON THE OBLIQUE-ANGLED TRIGON.

IF we describe a circle round the oblique-angled trigon A B C, and join the centre O with the three corners, the angles A O B, B O C, C O A are, according to Lesson LI., Part I., doubles of A C B, B A C, C B A. And if perpendiculars *O D*, *O E*, O F be drawn to the sides, these sides

are bisected. Now B D is the sine of the half of the arc B C, that is the sine of the arc which measures the angle B A C, or as we say for shortness, is the sine of the angle B A C; C E in the same way is the sine of the angle C B A; and A F is the sine of the angle A C B. But the ratios of the halves of any magnitudes are the same as the ratios of the whole, wherefore the sides of a trigon are proportional to the sines of the opposite angles.



This theorem enables us to make many calculations concerning the angles and sides of trigons; but at once we meet with this obstacle;—a trigon may have one of its angles obtuse, the arc which measures that angle must be greater than a quadrant, and we have not yet seen anything of the sines of such arcs; indeed in hardly any of the trigonometric tables are arcs greater than a quadrant mentioned. We must then proceed to look into this matter.

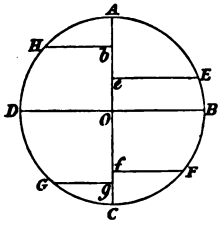
LESSON V.

ON THE SINES OF ARCS MORE THAN A QUADRANT.

Let the circumference of a circle be divided into four quadrants by the perpendicular diameters A O C, B O D, and let the arc be counted from A in the direction A B C D. When that arc as A E is less than a quadrant its sine is the perpendicular *e* E drawn from E to the radius O A, and its cosine is the distance O *e*.

While the arc A E increases to become a quadrant the point E moves up to B, and *e* to O; so that the sine of the complete quadrant is radius and the cosine is zero.

When E passes into the second quadrant B C, as to F, the sine Ff begins to decrease, and the cosine Of lies on the opposite side of O; and we see that the sine of A F is just equal to that of F C the *supplement* of A F or its defect from a half circumference. As to the cosine Of, we perceive that it is the sine of B F; but B F is not the *complement* of A F or its defect from a quadrant; on the contrary, B F is the excess of A F above the quadrant; the word *complement* is thus strictly inapplicable; however, we should meet with great inconvenience by changing the designation, and therefore we continue to use the word in the sense that B F is the *subtractive complement* of A F, or what must be subtracted from it in order to get a quadrant; and at the same time we hold Of to be the *subtractive cosine* of A F, or the cosine on the wrong side of O. In this we say sin A F = fF, cos A F = − Of. While F moves on to C, the sine fF decreases to become zero, and the cosine Of increases in the subtractive direction to become O C the subtractive radius, that is sin 180° = sin 200^g = 0; cos 180° = cos 200^g = − 1.



If the arc continue still to increase, its extremity passes into the third quadrant C D; the sine gG appearing on the opposite side of the diameter, so that − gG (that is gG to be subtracted) is held as the sine of the arc A B C G. The cosine of this arc is − Og. As the arc A G augments, its sine also augments in the subtractive direction, while its cosine grows less subtractive; and when G comes to D, the sine becomes − O D, and the cosine becomes zero. Lastly arcs, such as A B C D H, *greater* than three quadrants have their sines b H *subtractive, their* cosines O b additive.

This arbitrary extension of the meanings of words gives great convenience in our future explanations, by saving the use of long statements. The student will observe that by additive or subtractive we merely mean that the lines are counted in the usual or in the opposite direction; the directions for the first quadrant being held as the usual ones.

Since no angle of a trigon can exceed a half turn, the solution of oblique-angled trigons can only require the use of arcs less than the semicircumference.

Exercise 1.

Extract from the tables the sines and cosines of the following arcs: viz. 22° 14′; 71° 37′; 126° 19′; 163° 11′; 215° 22′; 261° 48′; 277° 13′; 351° 16′, and represent them also in diagram.

Exercise 2.

Draw and write down the sines and cosines of the following angles: viz. 8° 44′; 53° 41′; 98° 12′; 157° 46′; 196° 49′; 262° 47′; 288° 23′; 324° 38′.

The comparison of these two sets of examples shows that a sine or a cosine belongs to two distinct arcs. Thus two arcs whose sum is 180° or 540°, have the same sine; two arcs whose sum is 360° have the same cosine. Hence when an arc is to be determined by its sine or by its cosine alone, there is a doubt as to which of two arcs is to be taken; to decide between them requires some discriminating character derived from the conditions of the problem.

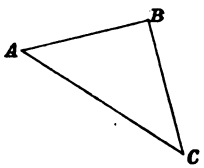
Exercise 3.

Write down the double values of the arcs a, b, c, d, to satisfy the conditions: $\sin a = +\ \cdot 57667$; $\sin b = -\ \cdot 63518$; $\cos c = +\ \cdot 75088$; $\cos d = -\ \cdot 66262$.

LESSON VI.

ONE SIDE AND TWO ANGLES OF A TRIGON BEING GIVEN, TO COMPUTE THE OTHER SIDES.

WHEN the angles of a trigon are given, the shape and the proportions of the sides are determined, so that when the length of one of these sides is known the others may be computed. Thus if, in the trigon A B C, the angle at A be 73° 23′, and that at C 64° 17′, the angle at B must be 42° 20′ and the three sides must be proportional to the sines of these three angles. Hence if the length of A C be 5739 inches the lengths of C B and B A are to be had by the proportions



sin C B A : sin B A C :: A C : C B

or ·6734427 : ·9582394 :: 5739 : C B,

sin C B A : sin B C A :: A C : B A

or ·6734427 : ·9009508 :: 5739 : B A.

Which give C B = 8166, B A = 7677·8.

There are several matters to be attended to in the management of these and similar calculations. In the very first place we have to consider the degree of precision that is sought for or that is attainable. The example before us implies that the side A C has been measured to the nearest inch; it may have been a little less or a little more than 5739 inches; or, strictly speaking, it is between 5738½ and 5739½ inches. Therefore to compute to any greater degree of accuracy than this is not merely useless, it is impossible. Now to get accuracy to four places we do not *need seven-place* trigonometrical tables, five places are

quite enough, so that we may shorten our calculations by writing

$$\cdot 67344 : \cdot 95824 :: 5739 :$$
$$\cdot 67344 : \cdot 90095 :: 5739 :$$

If, however, the base A C had been measured with greater precision, say to the nearest hundredth part of an inch, in which case it should have been written 5739·00, the whole seven places of the tables would have been needed.

In the second place we may consider the manner of working out the calculation. According to the mode shown in many books on arithmetic, the operation for the former of the two proportions would be set down thus,

$$\begin{array}{rrr}
 & \cdot 9582394 & \\
 & 5739 & \\
\hline
 & 8\cdot 6241546 & \\
 & 28\cdot 747182 & \\
 & 670\cdot 76758 & \\
 & 4791\cdot 1970 & \\
\hline
.6734427 & 5499\cdot 3359166 & 8166 \\
 & 5387\cdot 5416 & \\
\hline
 & 111\cdot 79431 & \\
 & 67\cdot 34427 & \\
\hline
 & 44\cdot 450046 & \\
 & 40\cdot 406562 & \\
\hline
 & 4\cdot 0434846 & \\
 & 4\cdot 0406562 & \\
\hline
\end{array}$$

But the numbers set down in the tables are not absolutely correct; they are only true to the nearest figure in the last place, having been computed to several places farther and then shortened. Hence on multiplying the sine of 73° 23′, which is ·9582394, by 9 we are by 10

means sure that the final figure of the product should be 6, nor, in multiplying it by 5 are we certain that the final figure should be 0; so that the last four figures 9166 are doubtful. If we be uncertain as to the 9, it is of no use to set down the figures 166; and therefore we may look for some mode of avoiding the useless labour of computing them.

Instead of beginning to multiply by the units of 5739, we may begin with the thousands and proceed downwards to the hundreds, tens and units, leaving off a figure at each step; only taking care to bring up whatever has to be carried. In this way the multiplication takes the form

$$\begin{array}{rr} & \cdot 9582394 \\ 5739\cdot & \\ \hline 4791\cdot & 1970 \\ 670\cdot & 7676 \\ 28\cdot & 7472 \\ 8\cdot & 6242 \\ \hline 5499\cdot & 3360 \end{array}$$

Also in performing the division we do not annex 0's so as to extend the number of places, the digits of which are unknown, but we prefer to leave off a figure from the divisor at each step, only noting what is carried; thus

$$\begin{array}{rrl} \cdot 6734427 & | \; 5499\cdot 3360 & | \; 8166\cdot \\ & 5387\cdot 5416 & \\ \hline & 111\cdot 7944 & \\ & 67\cdot 3443 & \\ \hline & 44\cdot 4501 & \\ & 40\cdot 4066 & \\ \hline & 4\cdot 0435 & \\ & 4\cdot 0407 & \\ \hline \end{array}$$

EXERCISE 1.

The base A C being 1653 feet, and the angles B A C, A C B being 75° 45′ 00″ and 79° 36′ 40″ respectively, required the sides C B and B A.

EXERCISE 2.

E D F = 50° 02′; D F = 1015·2 feet; D F E = 96° 44′; required F E and E D.

EXERCISE 3.

H G I = 83° 38′ 25″; G I = 2075 feet; G I H = 94° 58′ 45″; required I H and H G.

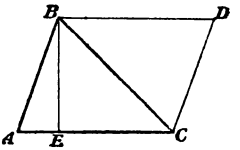
EXERCISE 4.

L K M = 61° 35; K M = 2839 feet; K M L = 62° 21′; required M L and L K.

LESSON VII.

GIVEN ONE SIDE AND TWO ANGLES, TO COMPUTE THE AREA OF A TRIGON.

IF the side A C be given, and two of the angles we can, as above, compute the side A B. Having let fall the perpendicular B E, we may also thence compute its length, since A B E is right angled at E. For these calculations we use the two proportions



sin A B C : sin A C B :: A C : A B
R : sin B A C :: A B : B E

which when compounded by multiplication become

R . sin C B A : sin A C B . sin B A C :: A C : B E

whence

$$BE = AC \times \frac{\sin ACB \,.\, \sin BAC}{R \,.\, \sin CBA},$$

but the area of the trigon is half the rectangle under AC and BE, wherefore

$$\text{Area } ABC = \tfrac{1}{2} AC^2 \times \frac{\sin ACB \,.\, \sin BAC}{R \,.\, \sin CBA}.$$

EXERCISE.

Compute the areas of the four preceding trigons.

LESSON VIII.

GIVEN ONE SIDE AND TWO ANGLES, TO COMPUTE THE DIAMETER OF THE CIRCUMSCRIBING CIRCLE.

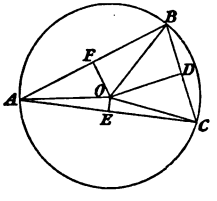
THE angle AOE is equal to ABC, but OA : AE :: R : sin AOE, wherefore

sin CBA : R :: AE : AO :: AC : diam. circ. circle

or

$$\text{diam. circ. circle} = \frac{AC}{\sin CBA}.$$

Thus it seems that we need to know only the angle opposite to the given side in order to compute the diameter of the circumscribing circle.



The division by sin CBA may be changed into a multiplication, for in the right-angled trigon AOE we have

R : cse AOE :: AE : AO :: AC : diam. circ. circle

diam. circ. circle = AC . cse CBA.

Thus division by the tabular sine of an angle may always be converted into multiplication by the tabular cosecant. Also division by the cosine of any angle may be turned into multiplication by the secant; advantage may be taken of this in the solution of the two preceding problems.

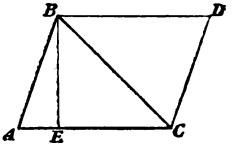
EXERCISE.

Compute the diameters of the circles circumscribing the same four trigons.

LESSON IX.

TWO SIDES AND THEIR INCLUDED ANGLE BEING GIVEN, TO COMPUTE THE THIRD SIDE AND THE OTHER ANGLES.

IF the two sides B A, A C and the contained angle B A C be known, we may divide the trigon into two right-angled trigons by drawing the perpendicular B E. In the one of these, A B E, we know the hypotenuse A B and the angle at A, and so may compute B E and A E. Knowing now A E and the side A C we may get E C; wherefore in the second trigon B E C, knowing B E and E C, the two sides of the right angle, we can find the hypotenuse B C and the angles.



When the angle B A C is obtuse, the perpendicular B E falls on the prolongation of C A, so that A E is on the unusual or *subtractive* side of A; and E C instead of being the *difference* between A C and A E, becomes their *sum*. This is in accordance with what has been said in

Lesson V. as to the cosine of an arc greater than one but less than two quadrants.

Since $BE = AB \cdot \sin BAC$ it follows that the area of the trigon is

$$ABC = \tfrac{1}{2} \cdot AB \cdot AC \cdot \sin BAC.$$

EXERCISE 1.

The angle BAC is $46° 23' 50''$; BA is 87 inches; AC, 76 inches; compute BC and the angles at B and C, first by letting fall the perpendicular from B, then by letting fall a perpendicular from C; compute also the area.

EXERCISE 2.

The two sides 387 and 493 contain an angle of $117° 43'$, required the subtense, the other angles and the area of the trigon.

EXERCISE 3.

An angle of $120°$ is contained by the two sides 192 and 133, required the subtense, the other angles and the area of the trigon.

EXERCISE 4.

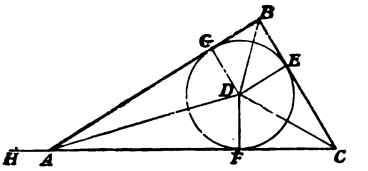
An angle of $60°$ is contained by the sides 491 and 234, required the subtense, the other angles and the area.

LESSON X.

GIVEN THE THREE SIDES, TO COMPUTE THE ANGLES.

WE have seen already, in Lesson LV., Part I., how to compute the radius of the inscribed circle and also the distances AF, CE, BG. Now in the right-angled *trigon* AFD we have the proportion $AF : FD :: R :$

tan D A F, so that the angle D A F may be found by help of its tangent. In the same way the halves of A C B and C B A may be got; and the accuracy of the work may be tested by taking the sum of the three angles.



EXERCISE 1.

Compute the angles of the trigons mentioned in Lesson LV., Part I.

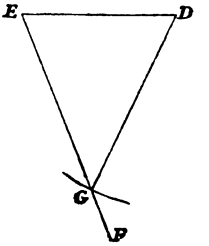
EXERCISE 2.

Compute the angles of the following trigons, viz. 785, 777, 748; 795, 763, 38; 816, 725, 469; 845, 442, 429; 904, 505, 409; 975, 970, 952.

LESSON XI.

GIVEN TWO SIDES AND AN ANGLE OPPOSITE TO ONE OF THEM, TO COMPUTE THE THIRD SIDE AND THE ANGLES.

THIS is to perform by calculation the solution of the problem contained in Lesson XVIII., Part I. If D E G be the given angle, E D and D G the given sides, the sine of the angle E G D may be got by the proportion G D : D E :: sin D E G : sin E G D.



Now we have seen that the same sine may belong to two arcs, one more and one less than a quadrant; wherefore, though the sine *be known, we remain* in doubt as to which of the two is

the arc wanted, and we have to remove that doubt by considering the conditions of the particular problem.

When the given angle is opposite to the longer of the two sides there is no doubt, because the angle opposite to the shorter side must necessarily be acute. Thus in the case E D = 5783, D G = 7089; D E G = 58° 47′, the angle at G must be less than D E G and so must be acute. The proportion in this case is

$$7089 : 5783 :: \cdot 8552135 : \cdot 6976583,$$

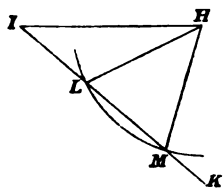
and we have to search in the tables for the arc whose sine is ·6976581 which is between 44° 14′ and 44° 15′, being almost exactly 44° 14′ 22″.

From this it follows that the third angle G D E is 76° 58′ 38″, and in order to find the remaining side G E we have the proportion

$$\sin D E G : \sin G D E :: D G : G E \text{ or}$$
$$\cdot 8552135 : \cdot 9742805 :: 7089 : 8075 \cdot 96$$

so that, counting only to the nearest unit, G E may be taken as 8076.

But when the given angle is opposite to the shorter of the two sides, the angle opposite to the longer one may be acute or may be obtuse. Thus if H I M = 38° 21′; I H = 8703; H M = 5937, the sine of the angle I M H is to be found by the proportion



$$H M : I H :: \sin H I M : \sin I M H \text{ or}$$
$$5937 : 8703 :: \cdot 6204636 : \cdot 909\,5325,$$

the fourth term of which is the sine of 65° 26′ 27″ or of 114° 33′ 33″; the former belonging to the further intersection M of the circle described from H with the prescribed radius; the latter to the nearer intersection L.

If the conditions of the problem with which these measurements are connected, be such as to discriminate between the two, we conduct our future calculations accordingly. Thus if the angle opposite to I H be to be obtuse we adopt the value I L H = 114° 33′ 33″, which gives L H I = 27° 05′ 27″; but if that angle be to be acute we take I M H = 65° 26′ 27″, giving M H I = 76° 12′ 33″.

We have seen that the solution of this problem is impossible when the proposed side H M is shorter than the perpendicular let fall from H to the line I K. The impossibility is exhibited in numbers when the computed sine of the angle at M is greater than unit. Thus if the case were proposed, H I M = 47°, I H = 783, H M = 546, we should have the proportion

546 : 783 :: ·7313537 : 1·048 etc.;

but the sine of no arc whatever can exceed the radius of the circle, and therefore no trigon can exist with these dimensions.

EXERCISE 1.

In the trigon A B C, A B = 59, B C = 78 and B A C = 98° 27′, compute the angles at B and C and the side A C.

EXERCISE 2.

E D being 382, D G 473 and the angle D E G being 73° 43′, compute the remaining angles and the third side E G.

EXERCISE 3.

The angle D E G being 62° 10′ 30″; D E being 6305 and D G 5674, compute the angles D G E, G D E and the side G E.

The computation may also be done by letting a perp-

dicular fall from D, and resolving the right-angled trigons thus formed.

EXERCISE 4.

Two sides being 625 and 221 and the angle opposite the latter being 20° 36′ 35″, compute the remaining angles and the third side.

Use also the perpendicular let fall upon the unknown side.

EXERCISE 5.

The side opposite to an angle of 48° 20′ being 5460, and one of the sides adjacent to it being 7309, compute the other angles and the third side.

EXERCISE 6.

The side opposite to 54° being 1431, and a side adjacent to it being 1769, compute the other angles and the third side.

LESSON XII.

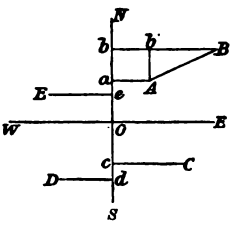
ON CO-ORDINATES.

IN order to mark the relative positions of a great many points, such as the towns and hills of a country, we make use of a map; and by reference to that map and to its scale of miles we find the distance between any two of them, and the direction in which the one is from the other. We have already seen that the precision of paper measurement falls far short of what is easily obtainable by help of numbers; it is then desirable to have some mode of representing the positions of points numerically. Maps are generally made on rectangular pieces of paper, and so the positions of points on it came naturally to be spoken

of in reference to the length and breadth of the map. Also, that part of the world which was best known to the Romans, lay along the coasts of the Mediterranean Sea, so that the length of the map lay from west to east, its breadth from north to south. Hence distances measured eastward or westward were called *Longitudes* (lengths), distances measured northwards or southwards were called *Latitudes* (breadths).

This method of indicating the positions of places by means of longitude and latitude, made use of by geographers, has been extended to geometry and land surveying.

If we draw through any point O two lines at right angles, as E O W which we may say is east and west, N O S, north and south; and if from some point A we draw A *a* perpendicular to one of them, say to N O S, the distance *a* A is called the *longitude* while O *a* is called the latitude of A.



In this way what is called the latitude of a point is its distance from the line W O E, while its longitude is its distance from N O S; and thus also we have north and south latitudes, east and west longitudes. The point D, for example has west longitude and south latitude, while A is in east longitude and north latitude.

It is very inconvenient to use the letters N and S for north and south latitude; E and W for east and west longitude, as is still done by sailors; it is much better to fix on one of the two directions as the usual one, and to indicate it by the sign + or by the want of a sign; the opposite or unusual *direction being* marked by the sign − .

We shall take north latitude and east longitude for the

usual directions, so that when we write lat A = 13 we shall mean that A is 13 units north of the line W O E, and when we write lat C = − 14 we mean that C is 14 units south of the same line.

When the latitude and longitude of a point are given its position may be marked. Thus if lat B = 21, lon B = 32, we shall get the position of B by measuring O b = 21 northwards, by drawing through b a parallel to O E and measuring on it b B = 32. In performing this operation O b used to be called the abscis (cut off) and b B the ordinate; but we might as well have measured an abscis of 32 along O E, and set up an ordinate there of 21. Wherefore we call the latitude and the longitude of a point its *co-ordinates*.

When we have to indicate the positions of several points, it is usual to write their co-ordinates in a tabular form, thus:—

Point.	Lat.	Lon.
A	13	12
B	21	32
C	− 14	21
D	− 20	− 17
E	10	− 22

The point O from which we reckon is called the ***origin*** (beginning) of the co-ordinates.

Exercise 1.

Mark the points A, B, C, D, E from the above data.

Exercise 2.

Mark, from the subjoined table, the points A, B, C, D, E, F, G, H; join by faint lines A B, B C, B F and by strong

lines A E, D H, C G, A D, E H, F G, E F, H G, D C. The resulting figure is an orthographic projection of a cube, B being supposed to be the farther corner.

	Lat.	Lon.
A	50	10
B	70	49
C	55	101
D	35	62
E	110	10
F	130	49
G	115	101
H	95	62

Exercise 3.

The adjoining table gives the orthographic projection of a regular octahedron, looking almost along the faces K M N, I O L.

	Lat.	Lon.
I	105	111
K	225	111
L	180	59
M	185	150
N	150	162
O	145	72

Exercise 4.

The following table gives another projection of the regular octahedron, to a different scale.

	Lat.	Lon.
I	20	60
K	140	60
L	100	21
M	95	112
N	60	99
O	65	8

Exercise 5.

From the subjoined table of the co-ordinates of its corners, draw the duodecagon A B C D E F G H I K L M.

	Lat.	Lon.		Lat.	Lon.
A	350	100	G	299	805
B	490	151	H	123	748
C	610	270	I	38	616
D	667	446	K	66	421
E	583	633	L	118	256
F	479	786	M	251	100

Exercise 6.

Project the points as given in the annexed table; join A B, B C, C D, D E, E F, F A; *a b*, *b c*, *c d*, *d e*, *e f*, *f a*; A *a*, B *b*, C *c*, D *d*, E *e*, F *f*.

	Lat.	Lon.		Lat.	Lon.
A	150	0	*a*	235	132
B	213	16	*b*	298	148
C	149	268	*c*	234	400
D	338	316	*d*	423	448
E	322	379	*e*	407	511
F	70	315	*f*	155	447

Exercise 7.

The accompanying table gives the projections of the corners of an oblong in perspective; the distance of the point of sight being ten inches from the paper.

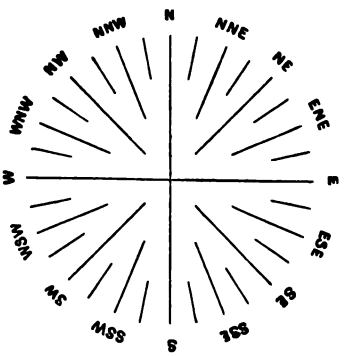
	Alt.	Lon.		Alt.	Lon.
P	−1·300	−2·000	*p*	+3·900	−2·000
Q	− ·956	−2·573	*q*	+2·868	−2·573
R	− ·807	+1·553	*r*	+2·422	+1·553
S	−1·040	+3·200	*s*	+3·120	+3·200

LESSON XIII.

ON DIRECTION, BEARING OR AZIMET.

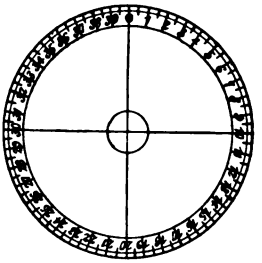
WHEN we are stationed at one place and wish to go to another, our first thought is as to the direction in which we must move. The word *direction* means straight away; astronomers and geographers often use the word *azimuth* with the same meaning, and seamen use the word *bearing*. Azimuth is Arabic, السمت azimet, as it should be written, meaning *intention to go.*

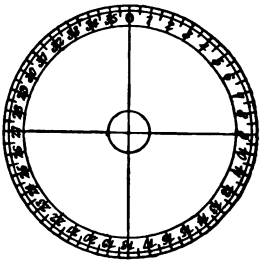
Sailors proceed by repeated halving, and divide the whole turn into 32 equal parts called *points*. These points are not numbered but are named according to a particular system. In the first place there are the four cardinal points north, east, south, west. The points midway between these are named NE, SE, SW, NW. The next bisections are marked NNE, ENE, ESE, SSE, SSW, WSW, WNW, and NNW respectively. Notwithstanding its great clumsiness, this system of names for directions is still in use by navigators, who also again divide the points into halves and quarters.

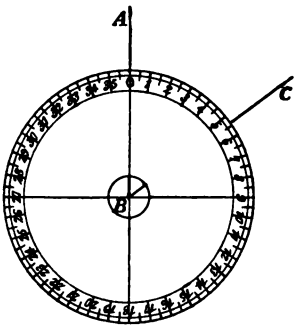


It is much more convenient to use the division of the circle into degrees, numbering these continuously from 0° to 360° or from 0ᵍ to 400ᵍ. If we agree to place the zero of the *protractor* always in one direction, say always *northward, the naming* of the degrees [illegible] indicate any

other direction, thus 90° or 100ᶜ will be toward the east, 180° or 200ᶜ toward the south, 270° or 300ᶜ toward the







west. Also 135° or 150ᶜ will be toward the south-east, and so on. The direction, bearing, or azimet of a line is thus the angle made by it and a line drawn due northwards; thus the angle b' A B is called the *bearing* of the line A B, and, in the language of the seamen B is said to *bear* from A so many degrees say 70°.

The bearing of A from B or the direction of B A being exactly the opposite, must differ from the direction of A B by half a turn, that is by 180° ancient or 200ᶜ modern.

Exercise.

Find, by the help of the protractor, the directions of the several lines in Exercises 1 and 5 of the preceding Lesson.

LESSON XIV.

GIVEN THE CO-ORDINATES OF TWO POINTS, TO COMPUTE THE LENGTH AND THE DIRECTION OF THE STRAIGHT LINE JOINING THEM.

If A and B (figure, p. 34) be the two given points, A b' is the difference between their latitudes or is lat B − lat A, while b' B is the difference between their longitudes or is lon B − lon A; wherefore the distance A B is given by the formula

$$A B = \sqrt{\{(\text{lon B} - \text{lon A})^2 + (\text{lat B} - \text{lat A})^2\}}.$$

Thus, if we take the co-ordinates as given in the specimen table, Lesson XII., lat B − lat A = 8, lon B − lon A = 20, wherefore $A B = \sqrt{\{64 + 400\}} = \sqrt{464} = 21{\cdot}54$.

When the ordinates have different signs their nominal difference becomes the actual sum: thus, referring to the example in p. 34, B is in north latitude 21, while C is in south latitude 14; the difference of their positions as to latitude is clearly 35, the sum of 21 and 14; and the distance B C is $\sqrt{\{35^2 + 11^2\}} = \sqrt{1346} = 36{\cdot}69$.

Again the points B and D are on opposite sides of the origin both as to latitude and as to longitude, the difference of their latitudes being 41 and the difference of their longitudes 49, so that the distance B D is $\sqrt{\{41^2 + 49^2\}} = \sqrt{4082} = 63{\cdot}89$.

Exercise 1.

Compute the length of each side of the duodecagon in the preceding Exercise No. 5, p. 36.

Exercise 2.

The positions of the points P and Q are lat P = 437, *lon P = 1623; lat* Q = 1022, lon Q = 3695, *compute their distance.*

Exercise 3.

The positions of the points R, S, T are lat R = 43, lon R = 113; lat S = 358, lon S = 33; lat T = 214, lon T = 341; required their distances.

If A b' were used as the radius of a circle having A for the centre, b' B would be the tangent of the arc measuring the angle b' A B, wherefore we have the proportion A b' : b' B :: R : tan b' A B or

lat B − lat A : lon B − lon A :: R : tan bearing of A B,

which may be written in the fractional form

$$\frac{\text{lon B} - \text{lon A}}{\text{lat B} - \text{lat A}} = \text{tan bearing A B.}$$

Thus in the present example,

tan bearing A B $= \frac{20}{8} = 2{\cdot}500\ 000\ 0$ whence the bearing of A B comes out 68° 11′ 55″.

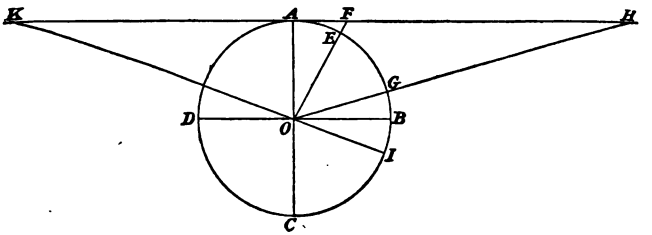
LESSON XV.

ON THE TANGENTS OF ARCS GREATER THAN A QUADRANT.

If we extend indefinitely both ways a line touching a circle at the point A, if from A we measure an arc A E, join O E and continue it to F, A F is what we call the tangent of the arc A E.

Let us now suppose the point E to creep along the circumference, accompanied by the prolonged radius O E F. As the arc A E augments, its tangent A F also grows greater, and when it comes to be A G nearly a quadrant the tangent *A H becomes* very long, so much so that, just as the

creeping point comes to B, the tangent has become, as we say, *infinite*, that is without an end, the line O B being then parallel to A H.



As soon as the moving point passes into the second quadrant B C, as at I, the radius O I must be continued in the opposite direction to meet the tangent line on the other side of A as at K. Thus the tangent of the arc A I is A K which being in a direction opposite to the usual one, takes the sign of opposition − . While the arc still continues to increase the *subtractive* tangent A K grows shorter and just when the moving point comes to C, the tangent becomes zero.

When the arc is greater than two quadrants the tangent appears again on the usual side of A, so that for all arcs between two and three quadrants it takes the sign + ; and lastly for arcs above three quadrants the tangents are the same as for those between one and two quadrants. Tangents with the sign + then belong to arcs in the first or third quadrant, tangents with the sign − belong to arcs in the second or fourth quadrant.

If we wish to compute the direction of the line A C in the preceding example, p. 34, we have tan bearing

$$A\,C = \frac{\text{lon } C - \text{lon } A}{\text{lat } C - \text{lat } A} = \frac{21 - 12}{-14 - 13} = \frac{9}{-27} = -\cdot3333$$

now ·3333 is the tangent of 18° 26' wh

belongs to 161° 34′ the supplement of the above; so that C *bears* from A 161° 34′, and A bears from C 341° 34′.

Thus when the co-ordinates of the points are known we can compute the direction in which to proceed from the one to the other and also how far we must travel. It is in this way that the navigator knows in what direction to steer his ship; only as he has to move on the spherical surface of the sea his calculations are somewhat more complicated.

Exercise 4.

Compute the bearing of P from Q, and that of Q from P, in Exercise 2, p. 39.

Exercise 5.

Compute the azimets of the lines R S, S T, T R of Exercise 3, p. 40.

Exercise 6.

The co-ordinates of the four points U, V, W, X, are lat U = 200, lon U = 70; lat V = 72; lon V = 570; lat X = 838, lon X = 790; lat W = 585, lon W = 994; required the length and directions of the six lines joining these points in pairs.

Exercise 7.

The co-ordinates of the five points A, B, C, D, E, being lat A = 175, lon A = 50; lat B = 243, lon B = 101; lat C = 323, lon C = 293; lat D = 179, lon D = 335; lat E = 79; lon E = 230; compute the lengths and bearings of the ten lines joining them in pairs.

LESSON XVI.

GIVEN THE DIRECTION AND LENGTH OF A LINE AND ALSO THE CO-ORDINATES OF ONE END, TO COMPUTE THE CO-ORDINATES OF THE OTHER END.

THIS question occurs daily in navigation; we know the latitude and longitude of the port of departure, the course of the vessel, the rate of sailing and the number of hours, and we wish to know where we now are. At present however our work is easier because we have to do with a flat surface, not with the curved surface of the earth.

The solution of the problem is quite simple; in the trigon A b' B of figure, p. 33, the angle b' A B is given and the length of the hypotenuse A B, wherefore A b' = A B . cos bear A B; b' B = A B . sin bear A B.

In order to show the application of this problem I shall suppose the point A to be in lat 200, lon 100; that from A in the direction 39° 58′ a line A B is drawn 137 long; from B in the direction 96° 44′ the line B C 145 long; from C in the direction 206° 47′, C D 233 long; and from D bearing 313° 23′ a line D E 175 long; and shall show how thence to compute the latitude and longitude of the extremity E of the last line.

The cosine and sine of 39° 58′ are ·7664183 and ·6423418 respectively; multiplying these by 137, the number of units in the distance A B, we get 105 for the change in latitude and 88 for the change in longitude, wherefore, adding these to the co-ordinates of the point A, we have lat B = 305; lon B = 188.

The direction 96° 44′ is in the second quadrant; now most *of trigonometrical* tables only go as far as to one quadrant.

and we therefore observe that cos 96° 44′ = − sin 6° 44′, while in 96° 44′ = cos 6° 44′; wherefore cos 96° 44′ = − ·1172485, sin 96° 44′ = + ·9931026; that is to say in moving from B to C the north latitude is diminishing, the east longitude is increasing. Multiplying these by 145 we get − 17 and + 144 for the changes in latitude and longitude, making lat C = 288, lon C = 332.

The direction of C D, 206° 47′ is in the third quadrant, and the cosine and sine are just the cosine and sine of 26° 47′ counted in the opposite directions; hence cos 206° 47′ = − ·8927169; sin 206° 47′ = − ·4506179; in moving from C to D the north latitude and the east longitude both decrease. Multiplying these by 233 we get − 208 for the change in latitude, − 105 for the change in longitude, whence lat D = 80, lon D = 227.

The direction of D E, 313° 23′, is in the fourth quarter; in moving from D to E, north latitude increases and east longitude decreases. The cosine of 313° 23′ is equal to the sine of 43° 23′, while its sine is equal to the cosine of 43° 23′, but counted in the opposite direction; hence cos 313° 23′ = + ·6868761, sin 313° 23′ = − ·7267745. Multiplying these by 175 we obtain + 120·2 for the change in latitude and − 127·2 for the change in longitude, wherefore lat E = 200·2, lon E = 99·8.

In this example the position of E agrees almost exactly with that of A; the measurements may be supposed to have belonged to a tetragon A B C D, and the slight difference between the positions of A and E as arising from the unavoidable errors of all measurements.

These calculations may be conveniently arranged as in the following scheme.

Point.	Bearing.	Distance.	Latitude.	Longitude.
A	39° 58′	137	200 + 105	100 + 88
B	96 44	145	305 − 17	188 + 144
C	206 47	233	288 − 208	332 − 105
D	313 33	175	80 + 120	227 − 127
A			200	100

In this example all the latitudes are North or + and all the longitudes East or + ; we may now see how the computations are to be managed when the co-ordinates happen to have various signs, as in the subjoined case, in which the calculations are made only to the nearest unit.

Point.	Bearing.	Distance.	Latitude.	Longitude.
A	8° 57′	6653	200 + 6572	200 + 1035
B	46 39	7045	6772 + 4836	1235 + 5123
C	140 53	6505	11608 − 5047	6358 + 4104
D	217 55	9593	6561 − 7568	10462 − 5895
E	245 05	7757	− 1007 − 3268	4567 − 7035
F	294 57	6697	− 4275 + 2825	− 2468 − 6072
G	322 05	9593	− 1450 + 7568	− 8540 − 5895
H	68 49	7829	+ 6118 + 2829	−14435 + 7300
I			8947	− 7135

Here the point D is in north latitude 6561 and the change from D to E is − 7568, that is 7568 southwards; wherefore the point E is in 1007 of south latitude as is shown by the sign − prefixed to 1007. From E to F the change in latitude is − 3268, wherefore the latitude of F is south 4275 or − 4275. Again E is in east longitude 4567 and the change is − 7035 or 7035 westward, wherefore F is in west longitude 2468 or as it is written in longitude −2468.

Exercise 1.

From the point A in lat 540, lon 100, A B is drawn in the direction 81° 30′ of the length 834 feet; from B, B C is drawn in the direction 170° 00′ of the length 483; from C, C D in the direction 257° 40′ is 686 long; and D A in the direction 334° 30′ is 553 long; compute the positions of the points B, C, D; and ascertain the amount of error in the new position of A.

Exercise 2.

The measurements being as in the annexed table, compute the co-ordinates of the several points, assuming lat A = 1000, lon A = 200. Compute also the lengths and directions of A F, C F, A H, G K and L E.

Station.	Bearing.	Distance.	Signal.
A	4° 36′	453	B
B	70 43	479	C
C	95 40	493	D
D	143 11	585	E
E	264 33	692	F
F	180 29	605	G
G	291 32	367	H
H	201 17	417	I
I	95 15	668	K
K	73 37	520	L

Exercise 3.

Lay down the adjoining measurements by help of the protractor and scale. Compute also the latitudes and longitudes, and lay down the figure by help of the co-ordinates, so as to compare the degrees of exactitude attainable by the two methods.

Station.	Bearing.	Distance.	Signal.
M	73° 50′	708	N
N	95 00	780	O
O	171 00	510	P
P	265 50	386	Q
Q	279 30	632	R
R	256 30	527	S
S	357 50	422	M

Exercise 4.

Assume, at will, the latitude and the longitude of T, compute the co-ordinates of the several points and ascertain the error, in latitude and in longitude, of the close.

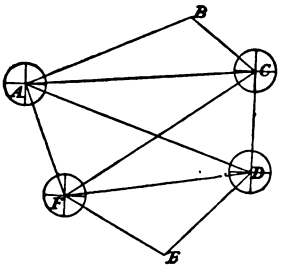
Station.	Bearing.	Distance.	Signal.
T	58° 15′	397	U
U	98 00	365	V
V	149 45	373	W
W	238 45	388	X
X	318 45	367	Y
Y	231 15	317	Z
Z	346 45	296	T

LESSON XVII.

TO ASCERTAIN, BY HELP OF THE PROTRACTOR, THE BEARINGS OF A NUMBER OF CONNECTED LINES.

LET it be proposed to observe the bearings of the straight lines in the adjoining figure connecting the points A, B, C, D, etc. In order clearly to understand the matter, we shall suppose that this figure represents part of a plan and that we are now about to do on paper what we shall afterwards have to do in the field. On paper we use the protractor, in the field we have the theodolite which is a protractor fitted with a telescope or other *sight*.

Fancy ourselves at the point A, and that we wish to get the direction of the lines A B, A C, A D, A F. We have first



of all to find out the north line; now the finding of the true north is a delicate and somewhat tedious operation, requiring some knowledge of the motions of the sun and stars, we shall have to spend one or two, or perhaps several days in order to accomplish it well. But though the true north be needed when we have to do with the map of a country, it is not required when we have only to do with the shape of a field; it is enough then, in the mean time, for us to assume some direction as a temporary north.

Having placed the centre of the protractor carefully over the point A, and turned its zero into the direction assumed for north, we note the degrees at each of the lines. Now *we cannot pretend to* guess to single minutes with a small

protractor; it will be enough for us to estimate the degree as divided into *six* parts each of *ten* minutes, or, perhaps, into *ten* parts each of *six* minutes. The bearings are, we shall say, A B 63° 50′; A C 81° 30′; A D 106° 40′; A F 155° 50′.

In order to get the directions of the lines proceeding from C, we must carry the protractor to that point; but we cannot now assume another north line; we must place the north line passing through C parallelly to that which was used at A. On paper we may draw the new north line by help of the parallel ruler; however the distance A C may stand for a few hundred yards and no parallel ruler can be used in the field, wherefore we must find some other way.

Now the direction of A C is 81° 30′ (not far from east); the direction of C A must then be nearly west, accurately 261° 30′, 180° different from the former. Wherefore if, keeping the centre of the protractor at C, we turn the instrument until 261° 30′ be in the direction C A, its zero line is parallel to the direction which it had at A, and the protractor is now in position to show the bearings of the other lines, as say C B 304° 10′; C D 177° 00′; C F 231° 00′.

Having finished the work at the point C, we may transfer the protractor to D. The direction of A D was 106° 40′ wherefore that of D A must be 286° 40′ and we may *set back* upon A by turning the instrument until 286° 40′ be upon the line D A. But we previously knew the direction of C D to be 177° 00′, wherefore, if all be right, that of D C should be 357° 00′; we therefore examine to see if it be so. Having satisfied ourselves on this point, we observe the bearings of the other lines and find them to be, say, D E 221° 10′; D F 257° 30′.

On carrying the protractor to F we may use any one of

the three *back-bearings* F A, 335° 50′; F C 51° 00′; F D 77° 30′ for placing the instrument in position, may verify it by means of the others and thereafter take the direction of F E, say, 114° 10′. In this way we have ascertained the direction of every line in the figure.

In performing this operation our principal aim is to keep the protractor in a position parallel, at each removal, to the position which it had at the first.

The most convenient form for recording these results in field work, is shown in the subjoined scheme. The first column contains the name or number of the *station* at which the theodolite is placed; the second column is for the observed bearing; the third column is left blank to be filled up afterwards from the chain field-book with the measured distance; and the last column is for the name, number or description of the *signal* looked at; which signal may or may not afterwards become a station.

Station.	Bearing.	Distance.	Signal.
A	63° 50′		B
	81 30		C
	106 40		D
	155 50		F
C	261 30		A
	304 10		B
	177 00		D
	231 00		F
D	286 40		A
	357 00		C
	221 10		E
	257 30		F
F	335 50		A
	51 00		C
	77 30		D
	114 10		E

When the bearings of two lines are known the angle which they make with each other is found by taking the difference between these bearings; thus BAC = 17° 40′; thus also the angle which BA and EF would make if produced to meet is 50° 20′, the difference between the bearings of AB and FE.

Exercise 1.

Construct a heptagon A B C D E F G to the dimensions B C = 58, C D = 33, D E = 29, E F = 50, F G = 41, G A = 52, A B = 39, B D = 85, D A = 92, A E = 75, E G = 73; and measure the bearings with the protractor, assuming A D to be north.

Compute the angles of all the trigons, thence get the bearings, and compare them with the measurements.

Exercise 2.

Construct the hexagon H I K L M N having H I = 169, I K = 125, K L = 208, L M = 156, M N = 187, N H = 205, H M = 356, M I = 385, I L = 317; placing I M towards the north, measure and also compute the bearings of the several lines.

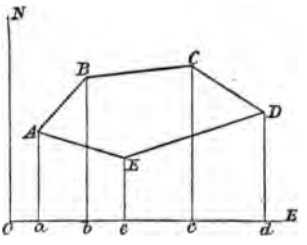
Exercise 3.

Placing P Q northward, measure and compute the bearings of the lines P Q = 363, Q R = 511, R S = 391, S T = 644, T U = 521, U P = 546, P R = 820, R T = 897, T P = 935.

LESSON XVIII.

THE CO-ORDINATES OF THE CORNERS OF A POLYGON BEING GIVEN, TO COMPUTE THE AREA.

If from the corners of the polygon ABCDE we draw the ordinates Aa, Bb, Cc, Dd, Ee, which are the latitudes,



we perceive that the area is the sum of the trapezoids aABb, bBCc, cCDd less the sum of dDEe and eEAa. Now the area of each of these trapezoids is easily computed, and so the required area may be found in any case. The surface aABb is equivalent to the rectangle under half the sum of aA and bB the two latitudes and ab which is the difference of the two longitudes or

$$2 \,.\, a\mathrm{AB}b = (\text{lat A} + \text{lat B})\,(\text{lon B} - \text{lon A}) \quad \text{also}$$
$$2 \,.\, b\mathrm{BC}c = (\text{lat B} + \text{lat C})\,(\text{lon C} - \text{lon B})$$
$$2 \,.\, c\mathrm{CD}d = (\text{lat C} + \text{lat D})\,(\text{lon D} - \text{lon C})$$

For the subtractive trapezoid dDEe we have the expression $\frac{1}{2}$ (lat D + lat E) (lon D − lon E) which may be put in the form

$$-2 \,.\, d\mathrm{DE}e = (\text{lat D} + \text{lat E})\,(\text{lon E} - \text{lon D}) \quad \text{also}$$
$$-2 \,.\, e\mathrm{EA}a = (\text{lat E} + \text{lat A})\,(\text{lon A} - \text{lon E})$$

and the sum of these five products (taking note of the signs + and −) gives twice the area of the pentagon.

In taking the details of the product of the sum of each *pair of latitudes* by the difference of the longitudes, we find

that the product of the latitude of a point by the longitude thereof occurs twice, once additive and once subtractive; thus lat A . lon A occurs subtractive in the first, and additive in the last of the lines. Hence all such products may be left out.

Setting ourselves to proceed round the polygon in the direction A B C D E A, and beginning with the point B, we find in the second line lat B . lon C additive and in the first line lat B . lon A subtractive, and for these two products we may write the single product lat B (lon C − lon A). Similarly we have lat C (lon D − lon B) and so on all round, wherefore

$$\begin{aligned} 2.\,ABCDE = &\text{ lat B (lon C} - \text{lon A)} \\ &+ \text{lat C (lon D} - \text{lon B)} \\ &+ \text{lat D (lon E} - \text{lon C)} \\ &+ \text{lat E (lon A} - \text{lon D)} \\ &+ \text{lat A (lon B} - \text{lon E)} \end{aligned}$$

that is to say, we multiply the latitude of each point by the longitude of the point before it *minus* the longitude of the point behind it, sum the products, attending to the signs + and −, and thereby get twice the area of the polygon; this law holds good whatever may be the number of corners and whatever may be the shape of the polygon.

The same formula may be arranged according to the longitudes, in which case we proceed in the opposite direction round the figure, thus

$$\begin{aligned} 2.\,BAEDC = &\text{ lon B (lat A} - \text{lat C)} \\ &+ \text{lon A (lat E} - \text{lat B)} \\ &+ \text{lon E (lat D} - \text{lat A)} \\ &+ \textit{lon D} \text{ (lat C} - \text{lat E)} \\ &+ \textit{lon C} \text{ (lat B} - \text{lon D)} \end{aligned}$$

An example of the arrangement of the calculation is subjoined. It is to be remarked that as each longitude is alternately additive and subtractive the sum of the numbers in the column marked "Diff. of lon." must be zero, attention being paid to the signs.

Point.	Lat.	Lon.	Diff. Lon.	Products +.	Products −.
A	362	182	− 17		6 154
B	537	325	+ 422	226 614	
C	583	604	+ 313	182 479	
D	425	638	+ 272	115 600	
E	412	876	+ 172	70 864	
F	194	810	− 218		42 292
G	270	658	− 365		98 550
H	143	445	− 316		45 188
I	301	342	− 263		79 163
			0	595 557	271 347
				271 347	
Area = 162 105				324 210	

In order to guard against mistakes it is well to use the second arrangement as a check.

Exercise 1.

Compute the area of the duodecagon, Exercise 5, Lesson XII. (Page 36.)

Exercise 2.

From the calculated co-ordinates compute the areas of the decagon given in Exercise 2, Lesson XVI., and of its several portions. (Page 46.)

Exercise 3.

Compute also the areas of the heptagons given in Exercises 3 and 4, Lesson XVI. (Page 47.)

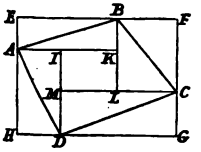
In the case of four-sided figures the calculation may be reduced to two products; thus the expression

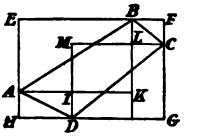
$$\begin{aligned} 2\,ABCD = &\text{ lat A (lon B} - \text{lon D)} \\ &+ \text{lat B (lon C} - \text{lon A)} \\ &+ \text{lat C (lon D} - \text{lon B)} \\ &+ \text{lat D (lon A} - \text{lon C)} \end{aligned}$$

may be written

$$\begin{aligned} 2\,ABCD = &\text{ (lat A} - \text{lat C) (lon B} - \text{lon D)} \\ &+ \text{(lat B} - \text{lat D) (lon C} - \text{lon A)} \end{aligned}$$

In the adjoining figure MI = lat A − lat C, and ML = lon B − lon D, so that the rectangle IKLM represents the first of the above products; also HE = lat B − lat D, EF = lon C − lon A, wherefore EFGH represents the second product, and it is clear that ABCD, being as much greater than the one of these rectangles as it is less than the other, is half their sum.

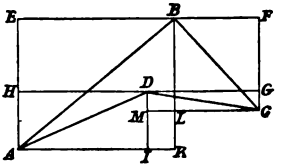


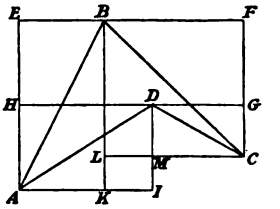


In the second figure MI is lat C − lat A or as we often put it − MI = lat A − lat C and the rectangle IKLM becomes subtractive, so that ABCD is half the difference of EFGH and IKLM; and it may be observed that the order of the letters is the reverse of what it was in the previous case when IKLM was additive.

When the tetragon has a reverse or re-entrant angle as in the *third figure, the* rectangle IKLM is detached from *EFGH, the tetragon* being in this case half the difference

of E F G H and I K L M; whereas in the fourth figure the tetragon is half the sum, and it may be noted that the order of the letters I K L M in the fourth figure is the





same as in the first figure. In this case – I M = lat A – lat C, – M L = lon B – lon D. Here both the difference of latitude and the difference of longitude have been reversed in direction, and the result is that the order of the letters is restored.

If, when arranging the calculation by the latitudes, we go round any polygon from left to right, the area comes out with the sign + ; on proceeding from right to left the area comes out with the sign –.

Exercise 4.

Compute the area of the tetragon A B C D when lat A = 316, lon A = 115; lat B = 365, lon B = 392; lat C = 166, lon C = 512; lat D = 83, lon D = 217.

Exercise 5.

What is the area of the tetragon E F G H when lat E = 218, lon E = 91; lat F = 416, lon F = 463; lat G = 324, lon G = 527; lat H = 119, lon H = 236?

Exercise 6.

When lat I = 128, lon I = 115; lat K = 403, lon K = 455; lat L = 187, lon L = 610; lat M = 238,

and lon M = 397; what is the area of the tetragon I K L M?

EXERCISE 7.

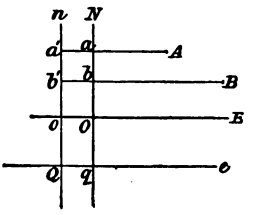
Lat N = 130, lon N = 100; lat O = 463, lon O = 295; lat P = 207, lon P = 576; lat Q = 307, lon Q = 410; required the area of N O P Q.

LESSON XIX.

ON THE TRANSPOSITION OF THE ORIGIN OF THE CO-ORDINATES.

VARIOUS nations reckon the longitudes of places from different meridians; thus the English count from the meridian of Greenwich, the French from that of Paris; and we have often to change from one mode of reckoning to the other. In the same way, for some particular purpose, it may be desirable to reckon the co-ordinates from another origin.

Let it then be proposed to change the origin from the point O to the point Q, keeping the axes of co-ordinates unchanged in direction.



Having drawn through Q the lines Q *n* Q *e* northward and eastward, that is to say parallel to O N and O E, and continued, if necessary, A *a* and B *b* to meet Q *n* is *a'* and *b'*. Q *a'* and Q *b'* are the latitudes of A and B, *a'* A and *b'* B their longitudes *in reference* to the new origin Q, and it *is clear that all the* latitudes are increas

as the case may be, by the same quantity Q o, while all the longitudes are changed by one quantity o O or Q q; Q o and o O being the latitude and longitude of the former origin O in respect of the new origin Q.

One use of such a transposition may be seen in this, that when computing the area of a polygon we may subtract one quantity from all the latitudes, so as to obtain smaller multipliers; or again that we may avoid the inconvenience of having co-ordinates on the − side, by adding a sufficient quantity to all the latitudes or to all the longitudes.

EXERCISE 1.

In the case of the duodecagon, Exercise 5, Lesson XII. (p. 36), place the new origin of co-ordinates at the point E.

EXERCISE 2.

In Exercise 2, Lesson XVI. (p. 46), place the origin at F.

EXERCISE 3.

In Exercise 3, Lesson XVI. (p. 47), make Q the origin of co-ordinates.

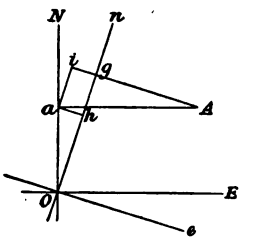
LESSON XX.

TO CHANGE THE DIRECTIONS OF THE AXES OF CO-ORDINATES.

PERHAPS, when a surveyor has had to begin a survey, there was no opportunity of getting the true north; but in the course of the work he has been able to find the error of his assumed direction. In this way he may need to refer the various points to a new set of axes.

Thus while ON was the assumed north, we may find

that O n is the true; and having already got the co-ordinates of many points in regard to the lines O N and O E, we may need to compute them in relation to the new lines O n and O e.



The problem then becomes this: knowing the latitude O a, the longitude a A, and the angle N O n of deviation, to compute the new ordinates O g and g A.

If we draw $a\,h$ and $a\,i$ parallel to O e and O n, and meeting O g and A g we form two right-angled trigons O ah and A $a\,i$ similar to each other, and we find O h = O a . cos N O n, $a\,i = h\,g = a$ A . sin N O n, wherefore the latitude of A according to the new system is

$$(\text{Lat A}) = \text{O}\,g = \text{lat A} \cdot \cos \text{N O}\,n + \text{lon A} \cdot \sin \text{N O}\,n,$$

and also since i A = a A . cos N O n, and $a\,h = i\,g$ = O a . cos N O n we have

$$(\text{Lon A}) = g\,\text{A} = -\text{lat A} \cdot \sin \text{N O}\,n + \text{lon A} \cdot \cos \text{N O}\,n$$

by help of which formulæ it is easy to compute the new latitude and the new longitude when N O n, the angle of deviation is known.

A change in the directions of the axes to which the ordinates are referred can make no change on the angles or dimensions of the plan, and therefore the bearings of the various lines must all be altered by the same angle N O n. For the position as shown in the figure, each bearing in the new system must be *less* than the corresponding bearing in the old system by the angle of deviation.

Seamen have constantly to attend to this

obtain the direction of the motion by means of the magnetic compass; but the magnetic north differs from the true north, by what is called the *variation*, wherefore the *observed* bearing must be corrected to get the *true* course of the vessel. Moreover, as we shall afterwards see, the variation changes from place to place and from time to time.

EXERCISE 1.

In the case of the tetragon A B C D, Exercise 4, Lesson XVIII. (p. 56), suppose the angle N *o n* to be 16° 15′ 37″ (of which the sine is ·28) compute the co-ordinates for the new axes.

EXERCISE 2.

For Exercise 5, let the axes be turned through an angle of 20° 36′ 35″ (of which the sine is ·352) backwards, so that the new may exceed the old directions, and compute the new co-ordinates.

LESSON XXI.

ON POSITION IN SPACE.

WE have been considering the position of a point on a flat surface, and must now proceed to consider the relative position of points not in one plane. For this purpose we fix upon a plane, say a horizontal one, and measure the distances of the various points therefrom. Thus we may take the level of the sea and ascertain how many feet each point may be above or below that level. For the complete determination of the position of a point in space we thus need *three* co-ordinates, viz. the *latitude*, the *longitude* and the *altitude*.

In order to get a clear idea of this subject, let us fix on some reference point O which we call the origin. Through O let a horizontal plane pass, called by surveyors the *datum level;* on this plane let two lines be drawn, one north and south, called the *meridian* line, and another at right angles to it east and west. Also at O let a line be drawn right upwards and downwards (toward the *zenith* and toward the *nadir*), that is in the direction of the plumb-line.

Imagine now one plane passing through the zenith, the nadir, the north and the south, which plane is called the *plane of the meridian;* and another passing through the zenith the nadir, the east and the west, called the *prime vertical* by astronomers. These three, the horizontal plane, the meridian, and the prime vertical, each extended indefinitely, divide the whole space round the point O into eight parts or right corners; and if from any point P, wherever situated, normals be drawn to these three planes, the lengths of these normals will serve to determine the position of the point. The distance from the prime vertical is the latitude; from the meridian plane, the longitude; and from the horizontal plane, the altitude of the point.

For convenience we shall reckon, as before, north latitude as +, east longitude as +; and we shall now consider altitude as + above the datum level, as − below it.

When the latitude, longitude and altitude of a point are given we may get its position by measuring the latitude along the meridian line, northward or southward as the case may be. Through the point so found we may draw a line eastward or westward equal to the longitude, and at the end of that line set up a normal equal to the altitude; the end of *that normal is* the point required. Or we may *take the three co-ordinates* in any oth

Exercise.

The co-ordinates of the six corners of an irregular octahedron being given as in the adjoining table, the student may set off the latitudes and longitudes on a board of soft wood and set up thin brass wires of the proper lengths, at the points so obtained; in this way he will show the position and form of the solid.

Point.	Lat.	Lon.	Alt.
A	10	12	53
B	67	14	23
C	73	93	21
D	8	89	60
E	51	43	98
F	27	59	2

LESSON XXII.

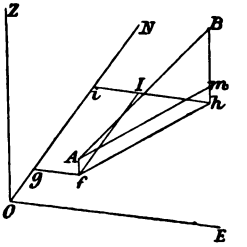
ON DIRECTION IN SPACE.

The *bearing* or *azimet* of an object is quite sufficient to tell us in what direction to look for it, if it be on a level with us. But when the object is not on a level, as in the case say of the sun or of a star, we must also know the angle of *elevation* or the *altitude* of the star as it is called by astronomers; also the object may be on a lower level in which case the angle is one of *depression*. When the bearing and the elevation or depression of an object are known, we easily form an idea of the direction in which to look for it. First we turn to the proper bearing, and then we raise our eyes to the known angle of inclination.

LESSON XXIII.

THE CO-ORDINATES OF TWO POINTS BEING GIVEN, TO COMPUTE THE LENGTH AND THE DIRECTION OF THE STRAIGHT LINE JOINING THEM.

O BEING the origin of the co-ordinates let O N, O E and O Z represent the lines drawn respectively northwards, eastwards and upwards; and suppose A and B to be two points in space.



From A let A f be drawn parallel to Z O to meet the plane N O E in f, then f A is the *altitude* of A. Draw now $f g$ parallel to E O and meeting O N in g, then $o g$ is the latitude, $g f$ the longitude of A. In the same way we get $o i$, $i h$, h B, the latitude the longitude and the altitude of the point B. These six ordinates are supposed to be known.

Having drawn $f l$ parallel to $g i$, $f l$ is the difference of latitude and $l h$ the difference of longitude. Join $f h$ and draw A m parallel to A, then m B is the difference of altitude. Now $AB^2 = Am^2 + mB^2 = fh^2 + mB^2 = fl^2 + lh^2 + mB^2$, wherefore

$$AB^2 = (\text{lat B} - \text{lat A})^2 + (\text{lon B} - \text{lon A})^2 + (\text{alt B} - \text{alt A})^2.$$

For example, if the positions of A and B be

lat A = 20; lon A = 15; alt A = 10;
lat B = 68; lon B = 51; alt B = 21;

we have
$$\begin{aligned}(\text{lat B} - \text{lat A})^2 &= 2304\\ (\text{lon B} - \text{lon A})^2 &= 1296\\ (\text{alt B} - \text{alt A})^2 &= 121\\ \hline \text{A B}^2 &= 3721\\ \text{whence}\quad \text{A B} &= 61\end{aligned}$$

The horizontal angle lfh is clearly the bearing of B as seen from A, while $m\,\text{A}\,\text{B}$ is the angle of elevation, but $fl : lh :: \text{R} : \tan flh$ and $\text{A}m : m\,\text{B} :: \text{R} : \tan m\,\text{A}\,\text{B}$, so that

$$\tan \text{bear A B} = \frac{\text{lon B} - \text{lon A}}{\text{lat B} - \text{lat A}} \qquad \text{and}$$

$$\tan \text{elev A B} = \frac{\text{alt B} - \text{alt A}}{\sqrt{\{(\text{lat B} - \text{lat A})^2 + (\text{lon B} - \text{lon A})^2\}}}.$$

In the present example $\tan$ bear $\text{A}\,\text{B} = \frac{36}{48} = \cdot 750$ $(\text{lat B} - \text{lat A})^2 + (\text{lon B} - \text{lon A})^2 = 3600$, whence $fh = 60$ and $\tan$ elev $\text{A}\,\text{B} = \frac{11}{60} = \cdot 1833333$ so that bear $\text{A}\,\text{B} = 36^\circ\ 52'$; elev $\text{A}\,\text{B} = 10^\circ\ 23'$.

Exercise 1.

The positions of the points A and B being lat A = 27, lon A = 15, alt A = 11; lat B = 75, lon B = 35, alt B = 50; compute the distance and the direction of B as seen from A.

Exercise 2.

The position of C being, lat C = 66, lon C = 47, lat C = 42, compute its direction and distance from each of *the preceding points.*

Exercise 3.

The ordinates of D being lat D = 21, lon D = 23, alt D = 110, required the directions and distances of A, B and C as seen from it.

Exercise 4.

The corners of the tetrahedron A B C D have the following co-ordinates, lat A = 81, lon A = 10, alt A = 0; lat B = 15, lon B = 106, alt B = 0; lat C = 130, lon C = 129, alt C = 0; lat D = 80, lon D = 80, alt D = 70; compute the directions and distances of D from A, B and C.

Exercise 5.

The corners of the pyramid A B C D E have the positions lat A = 10, lon A = 10, alt A = 0; lat B = 133, lon B = 19, alt B = 0; lat C = 122, lon C = 74, alt C = 0; lat D = 14, lon D = 68, alt D = 0; lat E = 70, lon E = 35, alt E = 72; compute the directions and lengths of A E, B E, C E, D E.

LESSON XXIV.

THE LENGTH OF A LINE, ITS DIRECTION AND THE CO-ORDINATES OF ONE END BEING GIVEN, TO COMPUTE THE CO-ORDINATES OF THE OTHER END.

For this purpose we must compute the differences of the co-ordinates. Now in the right-angled trigon A *m* B, Fig., p. 63, the hypotenuse A B is known, and also the angle of elevation *m* A B, so that we are able to compute *m B and A m or f h.* But again in the right-angled

trigon flh we know the azimet lfh and the hypotenuse fh, wherefore fl and lh may both be computed. Thus R : sin mAB :: AB : mB; R : cos mAB :: AB : fh or mB = AB . sin mAB, fh = AB . cos mAB; and again R : sin lfh :: fh : lh; R : cos lfh :: fh : fl, or $lh = fh$. sin lfh = AB . cos mAB . sin lfh; $fl = fh$. cos lfh = AB . cos mAB . cos lfh.

These results may be written thus:—

alt B − alt A = AB.sin elev AB,
lon B − lon A = AB.cos elev AB.sin bear AB,
lat B − lat A = AB.cos elev AB.cos bear AB.

As an example, let the position of the point A be lat A = 83; lon A = 68; alt A = 35; let the bearing of the line AB be 14° 15′ and its angle of elevation 47° 55′ 30″, while its length is 97; and let us proceed to compute the position of B.

We find in the table of sines, sin 47° 55′ 30″ = ·742 2683; cos 47° 55′ 30″ = ·670 1028, and multiplying each of these by 97 we get mB = 72; Am = 65; again sin 14° 15′ = ·246 1533; cos 14° 15′ = ·969 2309; these multiplied by 65 give lh = 16, fl = 63, wherefore

lat A = 83;	lon A = 68;	alt A = 35
+ 63	+ 16	+ 72
lat B = 146;	lon B = 84;	alt B = 107

While making surveys on uneven ground we have to measure the actual distances, such as AB; but in preparing the plan we can only mark the positions of the points f and h right below A and B; these points f and h are called the *projections* of A and B on the horizontal *plane, and* fh is called the *horizontal projection* of AB;

the length of fh is also called the horizontal distance of B from A.

Exercise 1.

In surveying an uneven field the adjoining results were obtained. Compute the co-ordinates of the several points, assuming the latitude of A to be 500, its longitude 200 and its height above the level of the sea to be 230 feet.

Alt.	Bearing.	Elevation.	Distance.	Sign.
A	37° 28′	+ 5° 17′	789	B
B	21 19	+ 2 41	903	C
C	83 48	+ 3 27	1023	D
D	157 03	− 1 41	978	E
E	217 13	− 4 15	817	F
F	253 51	− 4 19	963	G
G	287 05	− 1 01	823	A

Exercise 2.

The latitude of A being 40, its longitude 300 and its altitude 10, compute, from the annexed dimensions, the co-ordinates of the other corners of the octahedron. By help of these co-ordinates calculate the remaining sides of the solid and its three diagonals. Compute also the directions of D E, E A, A C, C D, D B, B F, F A.

From	Bearing.	Elevation.	Distance.	To
A	34° 28′	5° 37′	407	B
B	281 17	2 12	389	C
C	167 53	67 41	378	E
E	93 41	− 3 17	361	F
F	325 10	7 24	423	D

Exercise 3.

The positions of the corners of a trigonal hexahedron *given by the accompanying d*[illegible] Compute

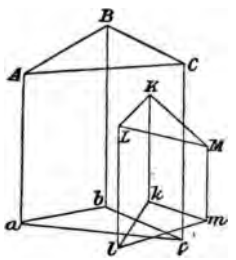
ordinates of H, I, K, L, assuming lat G = 150, lon G = 200, alt G = 20. Compute also the lengths of the remaining sides G I, G K, L I, L H, H K and of the diagonal G L.

From	Bearing.	Elevation.	Distance.	To
G	273° 14′	67° 29′	253	H
H	68 31	5 19	276	I
I	171 43	− 3 43	221	K
K	321 19	59 37	248	L

LESSON XXV.

TO COMPUTE THE SOLIDITY OF A POLYHEDRON WHEN THE CO-ORDINATES OF ITS CORNERS ARE GIVEN.

LET us imagine a flat-faced solid fixed above a horizontal plane, and from each of its corners let a plumb-line be drawn down to that plane so as to get the *plan* or horizontal projection of the polyhedron. The plumb-lines proceeding from the corners of one of the faces will delineate a prismoid resting on the horizontal projection and terminated by the face; and it is obvious that the solidity of the polyhedron is the difference between the sum of all the prismoids terminated by the upper surfaces, and the sum of all those terminated by the under surfaces of the solid. Also each face may be divided into trigons, so that the computation *may be resolved* into the calculation of the solidities of



three-sided prismoids. If A B C be one of the upper faces and *a b c* its horizontal projection, that projection is the cross section of a prismoid *a b c* A B C whose solidity is to be obtained (Lesson XXIII., Part II.) by multiplying the area of *a b c* by one-third part of the sum of *a* A, *b* B, *c* C the altitudes of the three points A B C.

Now if we proceed round *a b c* from left to right, its area is, according to what has been shown in Lesson XVIII.,

$$\tfrac{1}{2}\{\text{lat A (lon B} - \text{lon C)} + \text{lat B (lon C} - \text{lon A)} + \text{lat C (lon A} - \text{lon B)}\}$$

so that this multiplied by $\frac{1}{3}$ (alt A + alt B + alt C) gives the cubic contents of the prismoid.

Suppose now K L M to be on the under side of the polyhedron, *k l m* being its horizontal projection, then the solidity of the prismoid *k l m* K L M is to be subtracted. If then instead of proceeding round *k l m* from left to right, we go round it from right to left, the expression

$$\tfrac{1}{2}\{\text{lat K (lon L} - \text{lon M)} + \text{lat L (lon M} - \text{lon K)} + \text{lat M (lon K} - \text{lon L)}\}$$

comes out with the sign of subtraction, and when multiplied by $\frac{1}{3}$ (alt K + alt L + alt M) gives a subtractive result.

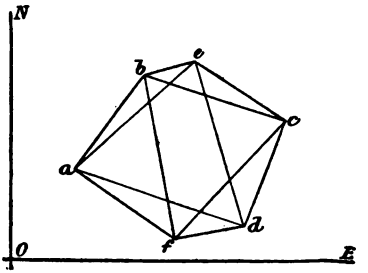
But if we fancy a fly to creep round the trigon K L M (feet upwards) from left to right, the accompanying progress round *k l m* will be from right to left. Hence if we divide the whole surface of the polyhedron into trigons, compute the area of the horizontal projection of each of these by the formula of Lesson XVIII., proceeding always from left to right, multiply that by the mean of the altitudes of its three corners, and take the aggregate of all these products, *paying attention to the signs*, we shall get *the solidity of the polyhedron.*

It is here to be observed that though some of the flat faces may have more than three sides, we must not take the entire area of the projection thereof and multiply that by the average altitude of its corners, but that we must divide each flat face into trigons and treat each of these separately.

As an example of this process we may take the octahedron having the co-ordinates as set down in the following table:—

Point.	Lat	Lon.	Alt.
A	120	83	74
B	244	178	193
C	183	364	182
D	45	310	79
E	262	243	63
F	30	220	170

For the purpose of getting a clear idea of the appearance and position of this solid, we may make its horizontal projection or plan as in the adjoining figure, and may imagine the various altitudes set up at the points *a*, *b*, *c*. . . . . From this we perceive that the trigon B C F forms as it were the upper face somewhat inclined to the horizontal plane, and that adjoining to this are three sloped faces A B F, F C D, C B E forming altogether the upper surface; while A D E, nearly horizontal, together with the three sloped faces *E B A*, *A F D* and D C E form the under surface of the



tahedron. If from the sum of the prismoids standing n bcf, fab, bec, cdf we subtract the sum of those tanding on afd, dce, eba, ade we shall get the solidity of the octahedron.

Now $2 . bcf$ = lat B (lon C − lon F) + lat C (lon F − lon B) + lat F (lon B − lon C) and therefore six times the prismoid B C F bcf is the product of this expression by the sum (alt B + alt C + alt F). We have in this case eight such products, namely one product for each trigon on the surface of the solid. On considering the whole attentively we perceive that one of the altitudes, say alt F, comes to be multiplied by each of the expressions for the projections of the trigons C F B, B F A, A F D, D F C; but the sum of these projections (having regard to the signs) is the projection $abcd$, and therefore the altitude of the point F has to be multiplied by the expression for this horizontal projection; hence if we take the product

alt F {lat A (lon B − lon D) + lat B (lon C − lon A)
+ lat C (lon D − lon B) + lat D (lon A − lon C)}

and also the analogous products for each of the other corners, the sum of the whole (having regard to the signs) is six times the solidity of the polyhedron.

This form of calculation is applicable to all kinds of flat-faced solids. Having divided the entire surface into trigons we imagine ourselves placed at one of the corners; then taking the adjoining ones in order from left to right we multiply the latitude of each of them by the longitude of the point before minus the longitude of the point behind, and having summed these products we multiply the sum by the altitude of the point at which we are *placed. The sum of* these products, taken for each corner.

attention being paid to the signs + and − , is six times the solidity of the polyhedron.

In the case of the octahedron the number of products is reduced; thus the above expression may be written

$$\text{alt F } \{(\text{lat A} - \text{lat C})(\text{lon B} - \text{lon D}) + (\text{lat B} - \text{lat D})(\text{lon C} - \text{lon A})\};$$

but the very same boundary, read in the opposite or − direction, must be taken when we are at the point E, and hence the two products may be conjoined thus:—

$$(\text{alt F} - \text{alt E}) \{(\text{lat A} - \text{lat C})(\text{lon B} - \text{lon D}) + (\text{lat B} - \text{lat D})(\text{lon C} - \text{lon A})\}$$

which with the two analogous products

$$(\text{alt B} - \text{alt D}) \{(\text{lat E} - \text{lat F})(\text{lon C} - \text{lon A}) + (\text{lat C} - \text{lat A})(\text{lon E} - \text{lon F})\}$$

$$(\text{alt C} - \text{alt A}) \{(\text{lat B} - \text{lat D})(\text{lon E} - \text{lon F}) + (\text{lat E} - \text{lat F})(\text{lon D} - \text{lon B})\}$$

makes up six times the solidity of the octahedron. In the above prescribed example these products become

$$(170 - 63)\{(120 - 183)(178 - 310) + (244 - 45)(364 - 83)\}$$
$$(193 - 79)\{(262 - 30)(364 - 83) + (183 - 120)(243 - 220)\}$$
$$(182 - 74)\{(244 - 45)(243 - 220) + (262 - 30)(310 - 178)\}$$

or

$$107\{63 \cdot 132 + 199 \cdot 281\} + 114\{232 \cdot 281 + 63 \cdot 23\}$$
$$+ 108\{199 \cdot 23 + 232 \cdot 132\} = 107 \cdot 64235 + 114 \cdot 66641 + 108 \cdot 35201$$
$$= 6873145 + 7597074 + 3801708 = 18271927$$

the sixth part of which gives 3045321 for the solidity of the octahedron.

Exercise 1.

Compute the solidity of the tetrahedron A B C D, and the lengths and directions of its six sides.

	Lat.	Lon.	Alt.
A	70	70	20
B	253	197	43
C	87	405	79
D	156	233	219

Exercise 2.

Compute the solidity of the trigonal hexahedron E F G H I, the lengths of its sides, and the length of its diagonal E I.

	Lat.	Lon.	Alt.
E	180	240	10
F	193	151	135
G	267	293	119
H	107	338	97
I	201	253	227

Exercise 3.

Compute the solidity of the octahedron in Exercise 2 of the preceding Lesson.

Exercise 4.

The point O is the apex and K L M N the base of a pyramid. In computing the solidity we may divi tetragon into two trigons either by the diagonal K M

L N, and, if the four points K, L, M, N be truly in one plane, the two modes must lead to the same result.

	Lat.	Lon.	Alt.
K	283	78	105
L	427	258	43
M	319	533	15
N	75	348	113
O	275	358	341

Exercise 5.

The two tetragons P Q R S and T U V W are connected by trigons contained by the lines P T, T Q, Q V, U R, R V, V S, S W, W P. Compute the bulk of the solid bounded by those faces, and show that P Q R S and T U V W are flat.

	Lat.	Lon.	Alt.
P	164	75	313
Q	249	168	217
R	219	417	63
S	154	525	17
T	286	188	143
U	484	473	97
V	226	638	203
W	79	458	373

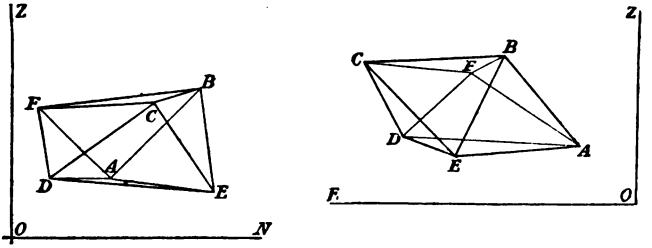
LESSON XXVI.

ON SIDE PROJECTIONS OR ELEVATIONS.

We obtain the plan or horizontal projection of an object by drawing through its points lines parallel to Z O. Similarly we may get its projection on the meridian plane Z O N by drawing parallels to E O; this would be called

by architects the *elevation* as seen from the east. In this elevation we use only the latitude and the altitude of each point.

Or again we may project the solid upon the prime vertical by drawing lines parallel to N O so as to give an elevation as seen from the north; and we may use these projections for the computation of the solidity.



If we were to look at the solid from the north the projection given in the third figure would have to be reversed; the reverse of the second figure would be a view from the west; and the reverse of the first figure would show the solid as seen from below, as in looking upwards at a roof. Thus a knowledge of the rectangular co-ordinates enables us to construct six views of the polyhedron.

This kind of projection, by lines drawn perpendicular to a plane, is called *orthographic.*

Exercise.

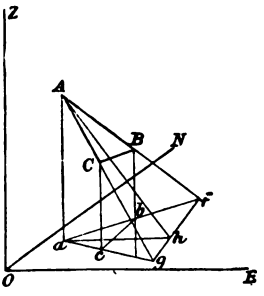
The student may draw the projections of all the solids mentioned in Lessons XXI. XXII. XXIII. XXIV. XXV. on the horizontal plane, on the meridian and on the pri vertical.

LESSON XXVII.

TO COMPUTE THE AREA OF A TRIGON FROM THE CO-ORDINATES OF ITS CORNERS.

WHEN the co-ordinates of the two ends of a line are known we may compute its length; wherefore, when the ordinates of the three corners of a trigon are given, the lengths of the three sides may be computed and thence the area. In this work the operation of extracting the square root occurs four times.

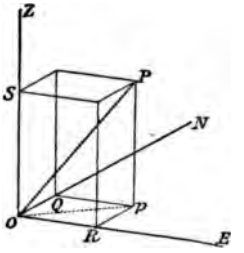
O N, O E and O Z representing the three axes of co-ordinates, let A B C be a trigon and abc its projection upon the plane N O E by the three lines A a, B b, C c drawn parallel to Z O. Continue A B, ab to meet in f, A C, ac to meet in g and join fg: this line fg is the common section of the two planes A B C, N O E. Draw A h perpendicular to fg and join ah, then (see Lesson IX., Part II.) ah is also perpendicular to fg, and the angle ahA measures the inclination of the two planes.



Since Af : A B :: af : ab, and Ag : A C :: ag : ac it follows that the areas of the trigons are proportional or that Afg : afg :: A B C : abc; but since trigons having a common base are proportional to their altitudes Afg : afg :: Ah : ah and again Ah : ah :: R : cos ahA, wherefore the area of the trigon A B C is to that of

its projection *a b c* as radius to the cosine of the inclination.

Suppose a line O P to be drawn from O in a direction normal to the plane of A B C, then, since O Z is normal to N O E, the angle Z O P measures the inclination of the two planes and is equal to the angle *a h* A of the preceding figure. If we now form an oblong having its faces along the co-ordinate planes, and having O P for its diagonal, the ratio of radius to the cosine of Z O P is that of O P to O S, so that the area A B C is to its projection on N O E as O P is to O S.



In the same way A B C is to its projection on N O Z as O P is to O R; and to its projection on Z O E as O P is to O Q. But the squares of O S, O R and O Q together are equivalent to the single square of O P, wherefore the square of the number expressing the area of a trigon is the sum of the squares of those which express its three projections; and hence

$$\{\text{lat A (lon B} - \text{lon C)} + \text{lat B (lon C} - \text{lon A)} + \text{lat C (lon A} - \text{lon B)}\}^2$$

$$+ \{\text{alt A (lat B} - \text{lat C)} + \text{alt B (lat C} - \text{lat A)} + \text{alt C (lat A} - \text{lat B)}\}^2$$

$$+ \{\text{lon A (alt B} - \text{alt C)} + \text{lon B (alt C} - \text{alt A)} + \text{lon C (alt A} - \text{alt B)}\}^2$$

$$= 4\,(\text{A B C})^2.$$

Exercise 1.

Compute the surface of the trigon A B C given by the accompanying scheme.

	Lat.	Lon.	Alt.
A	17	15	280
B	55	28	6
C	33	44	47

Exercise 2.

Compute the area of D E F from the accompanying ordinates. Find also the lengths of the three sides and from these again compute the area.

	Lat.	Lon	Alt.
D	17	23	43
E	91	7	17
F	51	55	4

Exercise 3.

The lengths of the sides and the area of the trigon G H I are all in integer numbers. Compute its area first from the ordinates, and then from the sides.

	Lat.	Lon.	Alt.
G	34	3600	84
H	4450	12	1924
I	1750	3912	799

Exercise 4.

Compute the surface of K L M from the accompanying co-ordinates.

	Lat.	Lon.	Alt.
K	59	58	465
L	431	219	53
M	112	480	21

The application of this method leads to only one extraction of a square root.

Knowing the areas of the three projections and also that of the trigon itself we may compute the cosines of the inclinations to the three co-ordinate planes.

The position or *lie* of a plane is known when the direction of a line normal to it is known; hence we have now only to determine the azimet of O P; but that azimet is the angle N Op and we have O Q : Qp or O Q : O R :: R : tan N Op. Now if O P be proportional to (or represent) the area of the obliquely placed trigon A B C, the lines O S, O R, O Q represent the areas of its projections upon the three planes, wherefore, O Q and Qp being known, the tangent of the azimet may be computed.

In general the direction of the normal to A B C is given by the formulæ

$$\text{tan bearing} = \frac{\text{projection on Z O N}}{\text{projection on Z O E}},$$

$$\text{sin elevation} = \frac{\text{projection on N O E}}{\text{area}}.$$

In order to distinguish that quadrant within which the direction falls, we must pay particular attention to the signs of the above algebraic expressions for the projections. When all the signs are + the direction of the normal is within the corner formed by the three planes E O N, N O Z and Z O E; when the projection on the horizontal plane comes out −, the normal is depressed; the direction in azimet depends on the two upright projections and must be determined according to the principle *down in Lesson XV.*

Exercise 5.

Compute the bearing and elevation of the normal to each of the surfaces A B C, D E F, G H I, K L M.

Exercise 6.

If the pentagon N O P Q R be divided into three trigons, and the directions of the normals to these be computed, they will be found to agree; this agreement shows that the five points are in one plane. If the solidity were computed ,according to the general formula, it would be found to be zero.

	Lat.	Lon.	Alt.
N	307	135	·7
O	10	341	13
P	60	263	42
Q	88	67	224
R	151	51	171

LESSON XXVIII.

ON VISION.

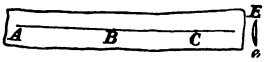
By far the greatest number of our measurements and observations are done by sight, and by sight also we are guided in our performances, and so it may now be time for us to study the powers and uses of the eye.

It does not seem very wonderful that, when some extraneous body touches us, we should be aware of the fact; but perhaps nothing in the world is more amazing than this: that by turning the eyes toward some object, *perhaps miles* away, we should instantly know of its

existence, of its colour, of its shape and almost of its size. By one glance at a tree we get information as to the arrangement of the branches and leaves, which could not be afforded by many folio volumes of tables of the co-ordinates.

We give the name *light* to that something which comes from the object to the eye. It is soon perceived that the object does not send out or *emit* light of itself, it has first to receive light from surrounding bodies, from the clouds or sky, or from the sun. Excepting the sun and stars almost all bodies shine or are visible by borrowed light.

Our first observation, or, to speak more correctly, our first idea, is that light comes to the eye in straight lines, and, in order to show this, we make a simple experiment. Having drawn carefully a straight line upon a deal board, let us stick into it three pins, one at each end and one about the middle of the line; then retiring to some distance and using only one eye, let us look along the row. We find that the nearest pin hides both of the others; now although quite convinced that the motion of light is rectilineal we cannot help perceiving that this kind of experiment is not conclusive, because the opening of the eye has considerable breadth, which may be represented by E*e* in the adjoining figure. Light, proceeding in straight lines from the farthest pin A, may pass on the one side or on the other side of the nearer pins B and C, and yet reach the eye, so that according to the *rectilineal* doctrine, the pin A ought not to be hid or covered by the nearer ones. This is really the case, so long as the thickness of the pins is less than the diameter of the eye-opening; but whenever that thickness is equal *to or greater than* the eye-opening, the farthest pin is



completely hid; so that this law of rectilineal motion is really confirmed by careful observation. Let us not be hasty; we shall have to measure the diameter of the eye-opening, the black circular spot in the middle. One person must do this for another: as soon as we begin, we find that the *pupil* rapidly changes its size; when the eye is turned towards a dark object, the pupil is large, when the eye is turned toward a bright light, the pupil becomes small. Hence it is hardly possible for us strictly to make the above experiment.

When there is plenty of light the pupil of the eye becomes small, so that less of the light enters; when there is little light, the opening widens so as to admit more of it. These changes are produced by an opaque coloured screen called the *iris*, the outer boundary of which does not change; the effect of light upon the iris seems to be to extend it and so cause it to shut the aperture; when there is little or no light the iris seems to contract to its natural size. In the case of nocturnal animals the expansion of the iris is very marked, as in the common house cat, the pupil of whose eye does not remain round but becomes a narrow slit in bright day-light.

On actually making the trial with a board a couple of feet long, we perceive all the three pins at once. When we attend to the pin at A, those at B and C are seen ill defined or blurred; when we look at B, A becomes blurred, C also being indistinct, and when we give our attention to C, both B and A are seen indistinctly. Moreover, we are quite sensible of an exertion in changing the attention from the one pin to the other; the eye can only be adapted to distinct vision for one distance at a time. If we take some small object in the hand, hold it at arm's length and *look with one eye*; then gradually bring it nearer and

nearer, we are conscious of an exertion in changing the eye so as to see the object distinctly, and when it is brought within a foot or ten inches the exertion becomes painful: within a smaller distance, which however is not the same with all individuals, we are unable, by any exertion, to obtain a distinct view.

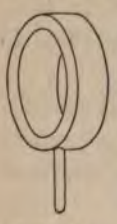
From this we perceive the difficulty of pointing a rifle to the target. The eye being set to see the target, cannot possibly see the vizé on the muzzle distinctly, still less can it see clearly the notch on the near sight.

In looking with only one eye, we turn that eye toward the object, and we are conscious of a movement in changing the attention from one point to a point extremely near it; as in examining first one side and then the other side of a small dot on the paper. This consciousness gives us an idea of the direction in which we are looking. The consciousness also of the adjustment to distinct vision may aid us in forming an idea of the distance of the object, because the one adjustment is associated with its particular distance.

We judge of the distance of an object more easily when we use both eyes; the directions in which these look, converge toward the thing looked at, and the degree of convergence, being always the same for the same distance, becomes associated therewith; so that the convergence of the two eyes and the adjustment of each to distinct vision go together; combined they produce the impression or idea of distance. We are unable to adjust the eye to distinct vision of a very near object, and we cannot accomplish the convergence of the two eyes when the object is close to the face.

When using both eyes we see only one object, yet really *we have two distinct* views thereof. The appearances of

a near object must be greatly different for the two eyes, since the right eye may see one side, the left eye the other side, so that with both eyes we may see both sides at once.



This may be well illustrated by holding a small hoop of, say, an inch in diameter and half an inch broad, in the hand so that its middle plane may be upright and directed to between the eyes. On looking attentively at the farther side, we notice distinctly the two ends as ovals nearly meeting at the middle, the nearer side of the hoop seeming to be much narrower than the farther side. But on transferring the attention to the near side, we see it grow apparently broader and the farther side contract.

From this it is clear that what we see is the compound of two views of the object, and that the compounding of these two views gives us the ideas of the distances, sizes, and arrangement of the parts of the object looked at.

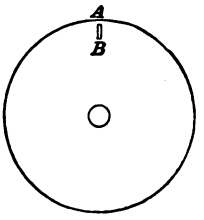
LESSON XXIX.

ON THE RAPIDITY AND DURATION OF SIGHT.

In good light, a glance at an object is enough to let us see it distinctly; the shortness of the time needed, may be shown by some simple experiments; thus, if one person write clearly a word or a few figures on a card, cover it up and then ask another person to read it when uncovered and again covered as quickly as possible, an exposure for a very small fraction of a second of time will be found to be *enough for* the easy reading. Or in a room lit by a single

flame, if we prepare a screen by which the flame may be completely hid, and remove that screen for as short a time as we may, all the objects in the room will be seen distinctly, or rather those of them to which the eye happens to be turned.

If we prepare a circular disc of stiff dark-coloured card with a pin-hole at the centre, so that it may be twirled on the end of a pin, and cut a narrow slit, as A B, near the edge; then bringing it close to the eye and causing it to rotate, we can only see remote objects while the slit is passing in front of the pupil of the eye. Using this little instrument, let us look at an object that is moving, say a man running or a horse at the gallop. We only get a sight of the object at each time that the slit comes round, and we see it at each time as if it were stationary or almost so; and in this way we are able to study instantaneous postures, blurred a little on account of the time occupied in the complete passage of the slit across the breadth of the pupil of the eye.



When it happens at night that a flash of lightning illuminates the air, we see any moving object toward which we may be looking, as quite stationary; the duration of the flash is, in general, so small that no perceptible change of place occurs while it lasts, and yet it is quite enough to give a distinct impression in the eye. This may be well shown by artificial lightning. Thus if we charge a Leyden jar by help of the electrical machine, place near it a wheel in rapid motion, the room being darkened, and then make the electrical discharge, we shall see the wheel q *stationary and* sharply defined. The dr

flash is so short as to be estimated with the greatest difficulty.

Although the impression be made thus quickly, it endures for some time. If we use the revolving disc with the slit, and look through it at some well-illuminated object, the successive views are perceptible when the disc turns slowly. When its speed is increased the impressions become more nearly continuous, and at a certain rate we fail to notice any interruption, that is to say the impression made on the eye at one passage of the slit continues until the slit come round again. It is generally stated that the impression from an ordinarily well-illuminated object lasts for one-tenth part of a second; that from a bright flame or from the sun lasts much longer.

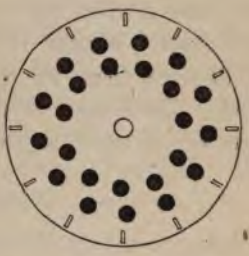
If we look at a moving body through the rapidly revolving slit we see as it were a succession of bodies placed at equal intervals, the interval being the distance through which the body has moved during one turn of the disc. In the case of an animal, as a horse or a bird, the postures are different according to the action at the instant of the passage of the slit; and the number of repetitions visible to the eye at once serves to measure the duration of the impression.

LESSON XXX.

ON THE MAGIC DISC.

If, instead of making one slit in the disc, we make a number of slits at equal distances, the same effects will be produced with a corresponding slower motion. Let a disc of this kind be prepared, dark on one face, with, say,

twelve slits round its edge; the centre being bushed with a common eyelet for the sake of durability; and let any figure consisting of twelve equal parts be drawn on the other face. Then if this, when rotating, be held before a mirror, and if we look at the image through the rim of slits, we shall see that image quite stationary, because each



view of it is an exact copy of the preceding view. But if the figure do not consist of twelve equal parts, each view will differ from that which precedes it, and there will be the appearance of change. Thus if we make a circular row of thirteen marks, each impression will appear slightly in advance of the preceding one, and the whole row will seem to move forwards at the rate of one division for each complete turn. But if we make a row of only eleven marks, these will seem to go backwards, because when the next slit comes to the eye, the next mark does not come quite to the place of the preceding mark. Thus it seems that, when the successive impressions do not differ much, each one instantaneously effaces the other, and that so the idea of progressive change or motion is caused. The student should carefully repeat these observations, making variations according to his own notions; he may easily succeed in producing a great variety of effects.

EXERCISE 1.

Make a disc of cardboard, say eight inches in diameter, with slits and a figure analogous to the first of the above figures.

EXERCISE 2.

In the open space between two slits and the arms of the star, draw a wheel rim with three spokes. In the next space draw a similar wheel with its spokes in a position 10° different from the preceding and continue this all round. The images of these wheels will appear to turn at the rate of one turn for each three turns of the disc.

EXERCISE 3.

Having constructed a disc with 12 slits, describe inside of these a circle, divide this into 11 times 13 equal parts and make a circular spot round each point of section. This row will appear to move slowly backwards, while the row of thirteen which may be put immediately inside of it, seems to move forwards.

EXERCISE 4.

Or if a wheel concentric with the disc be drawn with 13 spokes and 143 teeth, the spokes will seem to move one way, the teeth in the opposite way.

EXERCISE 5.

If the wheel be drawn with 11 spokes, the spokes and the teeth will seem to move in one direction but at different rates.

Without the instantaneousness of the impression we could not have got a distinct view of anything; because, *if some* perceptible time had been required, we should

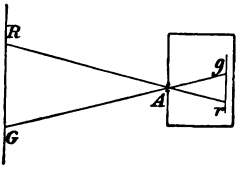
have had to keep the eye perfectly steady during that time, in order to prevent the view from being blurred. This is quite clearly shown by help of the revolving disc: each impression is made during the passage of a slit in front of the eye, this passage takes some small time, during which the figures to be looked at have moved, and in consequence all the views appear blurred at the edges, the indistinctness being greater near the edge of the disc.

The short endurance or *memory* of the impression seems to be needed in order that we may *attend* to the appearance of its parts, and the instantaneous effacement of one impression by the next prevents the confusion which would result from our being conscious of both at once.

LESSON XXXI.

ON THE DARK CHAMBER OR CAMERA OBSCURA.

THE light which reaches the pupil of the eye, enters within and there occasions those ideas or impressions of which we have been speaking. The eye resembles, in some degree, a room lit by a window: let us then examine how light comports itself in such a room. We may begin by darkening the room completely and then opening a small aperture in the window-shutter, say a gimblet-hole. On holding a sheet of white paper at some distance from the opening we perceive upon it a faint picture of the objects outside, in their proper colours but inverted in p red *point R outside sends* red light, part



the aperture A and reaches the white screen at *r* colouring it red; and so of some other point of a different colour, as green. In this way if a flat picture R G were set up outside with its plane parallel to the screen, the picture of it, *r g*, would, according to what has been shown in Lesson XXV., Part II., be similar to R G but inverted.

The student may easily find an opportunity of darkening some room all but a small opening, which he may close by a thin board. In this board he may make holes of different sizes and shapes.

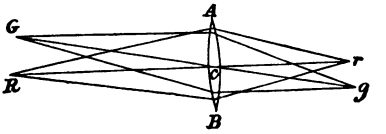
The opening at A must have some width; wherefore, supposing R to be a mere point, the light from it entering the aperture would fall, not upon a point at *r* but, upon a surface similar to the opening and somewhat larger in size, the scale of enlargement being in the ratio of R *r* to R A. Hence two closely contiguous points near R would be represented by two coloured surfaces at *r*, overlapping and confusing each other; and thus all the outlines on the screen *r g* must be blurred. In order to diminish this confusion or to get what is called *sharpness* in the *image r g*, we make the hole A very small; but in doing so we diminish the brightness, and with a very small hole the image becomes invisible. There are here two opposite conditions; to get brightness we need a wide opening; to get distinctness we require a small one. These conflicting requirements are reconciled by means of the *lens*.

LESSON XXXII.

ON THE LENS.

When a ball of glass, rock crystal, or other transparent substance has its surface polished, we see in it the images of surrounding objects much diminished and inverted; a hollow sphere of blown glass filled with water shows the same. Seal engravers used such balls for concentrating the light from the window or from a flame, upon their work. Between three and four hundred years ago it had occurred to some one that the roundness of the surface is the agent in producing these images, and that the intermediate mass of glass may be left away. This thought led to the formation of *lenses* or bits of glass shaped like the pea called *lentil* or *lens* in Latin; such a shape as is obtained by putting together two watch-glasses.

If we fit such a lens into the aperture of the dark room, bring the sheet of paper close to the lens, and withdraw it slowly, we find, at a particular distance, a bright and well-defined picture of external objects. It thus appears that the lens possesses the power of bending the path of light in such a way as that all rays leaving the point R and falling on the lens are converged to some point *r*, so that, when the screen is placed there, the image is quite sharp.



The lens permits us to use a wide aperture and yet to have all the sharpness which would [illegible] the use of *an exceedingly small one.*

A common spectacle eye may be fitted into the opening of the board of the dark chamber for the purpose of studying these appearances.

When, however, we examine the image very carefully, we notice that it is not quite sharp; we see a sharp image in a kind of haze. If we now prepare a circular disc of paper of the size of the lens, and cut out a circle of five-sevenths of its diameter from the middle, this *diaphragm* as it is called from the Greek διa through, and φραγμα a hedge, will cover about one-half of the surface of the lens, leaving only the inner half to transmit light. On applying this screen, the brightness of the image is reduced, but its sharpness is vastly improved; whereas, on removing the diaphragm and pasting the piece which had been cut out from it, upon the centre of the lens, the quantity of light remains as before but the indistinctness is much augmented, thus showing that the lens does not quite bend all the rays to meet at *r*, and that the error is greatest for the outer portions of its surface. However, on using a lens of only a moderate aperture, the performance is sufficient for many purposes and quite good enough for illustrating our present subject. Afterwards, when we shall have learned a great deal more concerning the properties and laws of light, we shall revert to this imperfection in the performance of lenses.

The action of the lens may be conveniently studied in a room artificially lit. Thus if we set up a lens as A *c* B and place a flame at R G, we shall find an image of that flame at some place *r g*, the scale of the image being less than that of the flame in the ratio of *r c* to R *c*; and if we now make the flame and screen exchange places, we *shall find* when the flame is at *r g*, its image formed on the

screen R G and magnified in the ratio of R c to r c. Also when the flame is brought nearer to the lens its image is farther removed; and if the flame be removed to a very great distance the image comes to within a certain fixed distance from it. This ultimate fixed distance may be ascertained by holding the lens to the sun's light and placing the screen so as to receive the sun's image. If the screen be of white paper the sun's image is dazzlingly bright, if the paper be of a dark colour it is much heated and may even be set on fire, hence the position of the image is called the *focus* (Latin, a fire-place), or the *principal focus* of the lens; while the two points R and r are called *conjugate foci* (yoked together).

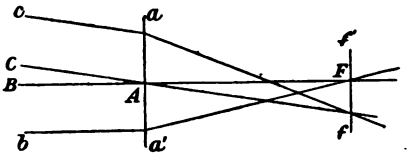
In order to form a lens such that all the rays leaving the point R should converge to r, it would be necessary to give to its surfaces certain peculiar forms not spherical. But we can only produce surfaces of sufficient accuracy by the process of grinding, and by grinding we can only get flat or spherical surfaces; hence we are unable to make lenses with what we know to be the true shape. The error caused by the use of spherical surfaces is called the *spherical aberration.* There is another and much more serious cause of haziness in this, that lights of different colours converge differently, thus giving rise to what is called *chromatic aberration* (χρωμα, colour).

These appearances may be studied conveniently in the darkened room. If we hold up a piece of thin tissue paper to receive the image made by the spectacle eye, examine that image through the paper by help of a pocket magnifying glass, and quickly withdraw the paper, we shall be able to examine the image made, as it were, in the air. The haziness and *the fringes* of colour may then be well seen.

LESSON XXXIII.

ON THE IMAGINARY PERFECT LENS.

To get a general understanding of the subject, we shall entertain the idea of a *perfect* lens; that is of a lens which would form a flat and perfect image of a flat distant object or painting placed before it. This lens may be supposed to be a thin flat circular disc $a\,A\,a'$, having its centre at A and its principal focus at F. In order that it may form a

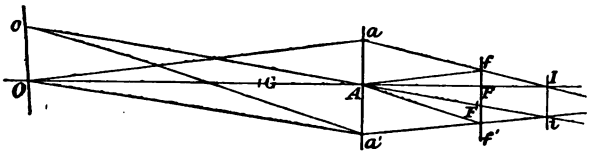


perfect image of a distant object on the plane $f\,F\,f'$ we must assume that any ray of light as C A passing through the centre A, is unchanged in direction and meets the focal plane at f; but that any other ray parallel to C A, such as $c\,a$, is bent at a into the direction $a\,f$; in which case all light parallel to c A and falling on the lens would converge to f; and, in the same way, all rays such as $b\,a'$ parallel to B A would converge to F.

The distance A F is called the *focal length* of the lens.

Here we have supposed the object from which the light comes to be so remote that the lines A C, $a\,c$ coming from one point in that object may be held as parallel, so that f is the image of that point. Let us now suppose the object to be brought to O o, O being in the axis of the lens, and let us inquire into where its image must be. A ray of

light proceeding from O to A passes onward in the direction A F; but a ray O a is bent at a. By drawing



A f parallel to O a to meet the focal plane in f, we get the direction af into which this ray is bent, so that, continuing af and A F to meet in I we obtain the point to which these rays converge, that is the image of the point O.

Since the trigon A F f is similar to O A a, we have O A : A F :: A a : F f; but again, because a A I is similar to f F I, A a : F f :: A I : F I, wherefore O A : A F :: A I : F I. Having measured A G equal to A F, we may write O A : A G :: A I : F I, and taking the difference of the first and second, as also the difference of the third and fourth terms, there results O G : A G :: A F : F I. Here we observe that G and F are the positions of the principal focus according as the light comes from the left or the right hand, and thus we may state that the focal distance for parallel rays is a mean proportional between the distances O G and F I of the object and image from the principal focus each on its own side; or the rectangle under O G and F I is equivalent to the square of A F.

The above proportion O A : A F :: A I : F I may be put in the form $\frac{AF}{OA} = \frac{FI}{AI} = \frac{AI - AF}{AI} = \frac{AI}{AI} - \frac{AF}{AI}$. But $\frac{AI}{AI} = \frac{1}{1} = \frac{AF}{AF}$, wherefore $\frac{AF}{AO} = \frac{AF}{AF} - \frac{AF}{AI}$; or, dividing all by AF, $\frac{1}{AO} = \frac{1}{AF} - \frac{1}{AI}$, or $\frac{1}{AO} + \frac{1}{AI} = \frac{1}{AF}$; that is

to say, the reciprocal of the focal distance for parallel rays is the sum of the reciprocals of the distances of the conjugate foci.

The very same proportions hold good for the figure o A F′ i a' drawn in order to find the point i conjugate to O, wherefore the image I i of O o is in a plane parallel to the lens.

When the distance O G is greater than G A, F I is less than A F and the image is smaller than the object; when O G is equal to G A, F I is equal to A F, and the image is of the same size with the object, the total distance between the object and its image being then four times the focal distance of the lens. If O G be made less than G A, F I becomes greater than A F, and the image is greater than the object. As O is brought closer to G, I moves rapidly off to a great distance, and F I becomes *infinite* when O is just at G.

Exercise 1.

In front of a lens having its focal length 20 inches, an object 3 inches in diameter is placed at the distance of 400 feet; required the position and size of its image.

Exercise 2.

The same object is brought to 300 feet, what is the change in position of the image?

Exercise 3.

For the distances 200 feet and 100 feet, what are the positions and sizes of the image?

EXERCISE 4.

Determine the positions of the conjugate focus, and the sizes of the image of the same object at the distances 90, 80, 70, ... 20, 10 feet respectively.

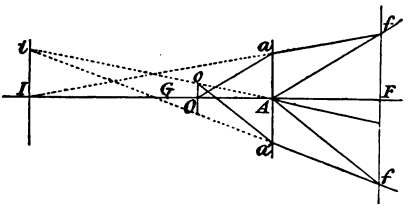
EXERCISE 5.

Compute the positions and sizes for the distances 110, 100, 90, ... 50, 40, 30 inches.

EXERCISE 6.

Make the same computations for the distances 28, 26, 24, 22, 21, 20½ inches.

We must now inquire what happens when the object is brought nearer to the lens than the principal focus, as to O in the adjoining figure. In this case F*f* is greater



than A*a* and the lines A F, *af* diverge; in order to meet they must be continued backwards in the directions F A I, *fa* I to I on the same side of the lens with the object O. Thus we see that all the light proceeding from O and passing through the lens, proceeds afterwards *as if it had come from* I; and, in the same way all the light leaving the adjoining point *o* proceeds, after passing through the lens, as if it had come from *i*, A *o i* being in one straight *line. Hence in looking* from the side F through

at the object O o we receive the light as if it had come from the image I i, and we imagine that there is an object there, larger than the actual object in the ratio of A I to A O, the image not being inverted.

In order to examine an object well, we bring it near to us; but the eye is unable to adjust itself to distinct vision for very near objects; however by introducing a lens a A a' between the eye and the object, we cause the light to enter the eye as if it had come from a more remote and larger object I i. The preceding formula $\frac{1}{AO}+\frac{1}{AI}=\frac{1}{AF}$ applies in this case also, provided that we notice the sign of A I to be $-$, as it is laid on the opposite from the usual side of the lens.

Exercise 7.

For the same lens of 20 inches focal length, compute the positions and sizes of the image of the object 3 inches across, when that object is placed at the distances 19, 18, 17, 16, 14, 12, 10, 8, 6, 4, 2 inches from the lens.

Exercise 8.

A lens of 1 inch in focal length is brought within the distance of ·9 of an inch from an object $\frac{1}{40}$ of an inch across; what is the position and what the size of the virtual image?

These remarks concerning the action of the supposed *perfect* lens, are in accordance with the performance of actual lenses, so far as the imperfection of that performance allows. The actual lens has thickness, and *when we speak* of its focal distance we use very inexact

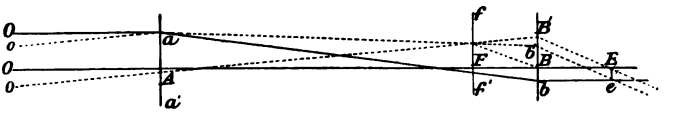
language; is the distance to be measured from the nearer surface, from the farther surface, or from midway between them? It is as when we say that such a village is so many miles from London. The language is well enough for general explanation.

By using the lens or *magnifying glass*, as we call it, we are able to examine minute objects with great advantage. Children should be made familiar with it; it is useful in taking a thorn out of the finger, or in examining the eye of a sewing needle.

LESSON XXXIV.

ON THE TELESCOPE.

When the light from some far off object $o\,O\,o'$ falls upon a lens A, an inverted image $f\,F\,f'$ of the object is formed at the focus F. This image would be visible on a screen of



paper placed at F, if the light falling on the screen from other objects were not too bright. In the dark, when the distant object is luminous, or in a dark room, the image is distinctly seen. Even in daylight we may see the inverted image by placing the eye in the line A F and so far off as to permit of distinct vision; the paper screen may, in that case, be removed, as the light will reach the eye just as if it came from an object $f\,F\,f'$. But we wish

to examine this seeming object $f\,\mathrm{F}\,f'$ minutely, and therefore we shall use a magnifying glass $b\,\mathrm{B}\,b'$.

If the second lens B be placed so that its principal focus coincide with F the principal focus of the lens A, the image $f\,\mathrm{F}\,f'$ formed by the first lens becomes an object (as it were) for the second lens, whose image is at an infinite distance. Thus all the rays O A, *o a*, coming from the remote point O and falling on the lens A *a*, converge to F, thereafter proceed to meet the surface of the second lens at B, *b*, and are there bent into parallel directions B E, *b e*, reaching the eye as if they had come from one point in a remote object.

The parallel rays *o* A, *o a* coming from some other point *o* on the remote object converge, after passing through the first lens, to the point *f* and proceed to meet the second lens at B', *b'*; they are there bent into the directions B' E, *b' e* parallel to *f* B, and reach the eye as if they had come from some very distant point in the direction E B'.

Hence the eye placed at E receives light through this combination of lenses, as if that light had come directly from an inverted remote object; but the breadth O *o* instead of subtending the angle $\mathrm{O\,A\,}o = f\,\mathrm{A\,F}$, appears to subtend the angle $\mathrm{B\,E\,F'} = f\,\mathrm{B\,F}$; and therefore appears magnified in the ratio of A F to B F. By this arrangement we are enabled to examine remote objects more advantageously, just as if we were brought nearer to them in the ratio of the focal distances of the two lenses. The two lenses are connected by a pair of tubes, one of which slides within the other for adjustment; the tubes serving to screen the eye from extraneous light; the whole instrument being called a *telescope*, from the Greek τηλε *far off*, and σκόπος *a looker*.

Exercise 1.

The object-glass of a telescope has a focal length of 3 feet; the eye-lens has its focal length $\frac{3}{4}$ of an inch; required the *magnifying power* of the instrument.

Exercise 2.

The moon subtends an angle of about 32′; what angle would it seem to subtend when viewed through this telescope?

N.B. The student must be careful to notice that the ratio of the arcs is not the ratio of their tangents.

It is clear from the construction of the figure, that if A a be regarded as an object in front of the lens B, E e is its image formed by that lens; all the light, from whatever source, passing into the telescope through the space A a proceeds ultimately through the much smaller space E e, so that if A a be the radius of the aperture of the lens, E e is that of a small circular space through which all the light coming through the telescope must pass; A a being greater than E e in the ratio of the magnifying power of the instrument. If the pupil of the eye be placed at E and if its diameter be more than the double of E e, the whole light from the telescope will enter the eye; but if the pupil be less than the space at E, which space is the image of the object-lens A, part of the light will be lost, and in reality only a portion of the lens a A a' will be utilized, the diameter of that portion being greater than that of the pupil in the ratio of the magnifying power.

Exercise 3.

The diameter, or as it is called the *aperture*, of the object-glass in the above telescope being 2·4 inches, required the position and width of the aperture at E through which all the light passes.

Exercise 4.

By how much must the tube be lengthened to suit an object at the distance of 100 feet, and what is then the magnifying power?

When we use the telescope to look at a somewhat near object, the image moves beyond F to the conjugate focus, and therefore we must provide the means for lengthening the tube sufficiently. When the distance is given we may compute the elongation, and conversely we may determine the distance by observing the elongation. This has been proposed as a means for measuring distance; it is not, however, susceptible of accuracy except for very near objects, for the which it is not needed.

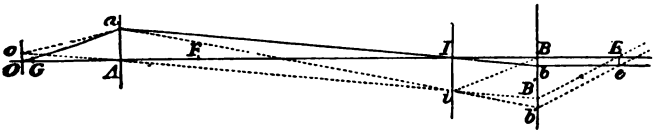
Telescopes which are to be used for surveying or astronomical purposes are provided with cross fibres, generally of spider-web, placed in the focal plane $f\,\mathrm{F}\,f'$. The coincidence of these with any particular point of the image shows to what exact point of the object the telescope may be directed.

The simple telescope, such as has been described, inverts; for the purpose of re-inverting the image we use what may be called a second telescope placed beyond the first.

LESSON XXXV.

ON THE COMPOUND MICROSCOPE.

THE compound microscope may almost be described as a telescope arranged for seeing near objects. The first or *object-lens* A of the microscope is made of short focal distance and the object O is brought near to the outer principal focus G, in which case the image is thrown to a considerable distance F I from the inner principal focus F, and is greater than the object in the ratio of A I to A O.



As in the telescope, this image is examined by means of a second or *eye-lens* B. By this arrangement we see a highly magnified and inverted image of the object. By placing cross fibres or transparent scales at I, we are able to measure the dimensions of the minute object, and the instrument is then called a micrometer.

We often hear the magnifying power of a microscope spoken of; some microscope is said to magnify one hundred times. Now it is quite easy to get a correct idea of the magnifying power of a telescope; the distant object O *o* is seen as subtending the angle O A *o* or F A *f*, but when looked at through the instrument it appears to subtend the angle F B *f* or B E B′; and if the lines E B, E B′ were continued as far as to the actual object, the image would appear greater than the object in the ratio of A F *to B F. The magnifying* power is ot

dimensions of the instrument; but for the magnifying glass and microscope it is not so. When we wish to examine a small object we bring it near to the eye, and our best view of it is got when it is as near as possible, say at a distance of ten inches. But if we use a lens of one inch focal length we are able to view the object as from a distance of only one inch, and so see it magnified, as it were, ten times. In this case a lens of *one inch* focal length may be said to have a magnifying power of *ten*; this is altogether comparative. A short-sighted person may be able to see an object so close as within eight inches of his eye; to him an inch lens would only have a magnifying power of eight times. The expression "*magnifying power of a microscope*" has thus only a conventional meaning, depending on what is assumed as the ordinary state of the eye-sight. It would be better to speak of the focal length of a magnifying glass or of what is called the *equivalent focal length* of a compound microscope.

The eye-lens B enables us to see the image I *i* as from the distance I B; but that image is already enlarged in the ratio of A I to O A; wherefore the object is viewed through the microscope as if seen from a distance which is a fourth proportional to A I, I B and O A, or as if seen through a single lens whose focal distance is $\frac{OA \cdot IB}{AI}$, which is the *equivalent* focal distance of the instrument.

Exercise 1.

A transparent scale divided to hundredths of an inch, is placed at the field-bar I of a microscope; the distance I A is 5 inches and the focal length of the object-glass is $\frac{1}{2}$ inch; required the distance A O at which the object must be placed, and the value of one division of the scale.

The eye-lens has a focal length of $\frac{3}{4}$ inch; required the equivalent focus of the instrument and its magnifying power in reference to an ordinary distance of 10 inches.

EXERCISE 2.

Substitute an object-lens of $\frac{1}{2}$ inch, and make the corresponding computations.

The student must be careful to note that these descriptions and investigations have been made supposing the lenses to act perfectly. In truth, no such lenses exist or can exist, and in practice we must be contented with close approaches to perfect action. The object-glass is made of two lenses, one of crown, the other of flint glass, arranged to correct both the chromatic and the spherical aberration; the eye-piece is also composed of two lenses, usually both of one kind of glass, arranged for the same purposes and also for enlarging the field of view. We shall have to discuss these points afterwards.

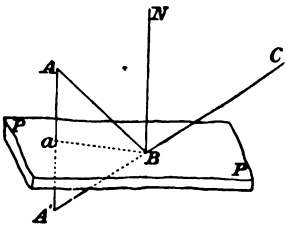
LESSON XXXVI.

ON THE REFLEXION OF LIGHT BY A FLAT SURFACE.

WHEN light falls obliquely upon a flat polished surface, it is reflected in a direction equally inclined to the surface; thus if A B represent a ray of light meeting the polished plane P in the point B, its direction, after being reflected, is to be found by raising B N normal to the plane at the point B, by imagining a plane to pass along the lines A B and B N, which plane is called the *plane of incidence*, and *by making therein* an angle N B C equal to A B N.

A B N is called the *angle of incidence*, N B C the *angle of reflexion*.

If we suppose A to be a bright point placed in front of the mirror P, the light proceeding from it and reaching the mirror must proceed as if it had come from a point A′ placed as far on the other side of the plane; for if we draw the normal A *a* and produce it to meet the continuation of C B in A′, the angle A′ B *a* being the complement of N B C is equal to *a* B A and consequently the distance *a* A′ is equal to *a* A. The eye, receiving the light which has come from A and been reflected by the mirror, receives it as if it had come directly from A′ which is therefore the image of the point A.

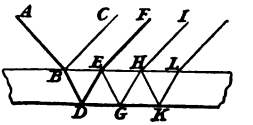


Hence if an object be placed in front of a mirror, the positions of the various points of its image are to be found by letting fall normals from them and producing these normals as far beyond. The form thus traced out is not *equal* to, it is *symmetric* or *twin* to the object; its solidity, if computed according to the formula by co-ordinates, would come out with the sign −.

If the end of a straight wire be brought to touch a mirror, the wire and its image form an angle double of the inclination of either of them to the polished surface: so when the wire is brought to be normal to the mirror its image appears as a continuation. If a carpenter keep his saw-blade, particularly the left face of it, bright, he is able to make a square cut, by holding the saw so as that *the image of the* edge of the piece of wood on which he is

operating, be in line with the edge. The surface of any liquid at rest forms a horizontal mirror, and if a plumb-line be held over it, the image is seen to be in continuation of the line. Also the image of the sun, as seen by reflexion from a fluid surface, is as much depressed below the horizontal line, as the sun itself is elevated above it.

The mirror or *speculum* used by the ancients was made of polished metal; the compound now used by opticians consists of copper, tin and a little arsenic, it is excessively hard and, unfortunately, very brittle. Mirrors or *looking glasses* for ordinary use are made of plate glass, which should be carefully ground flat and polished on both sides; one side is coated with a compound or *amalgam* of tin and quicksilver which gives a bright reflexion. But the looking glass is not simple in its action: the light on reaching the front surface of the glass is partly reflected and partly refracted or bent in entering the glass. That portion of it which enters is reflected by the silvered surface back to the front surface where it is again partly reflected in the glass and partly bent in escaping out of it. In this way there are two distinct images, one due to reflexion at the anterior surface, another much brighter formed by reflexion at the silvered surface. This is represented in the figure; the original ray A B is divided at B into a faint portion reflected along B C and a brighter portion refracted into the direction B D; this portion B D is reflected at D along D E which is divided at E into two portions, the brighter going out of the glass along E F and the fainter reflected into E G. This E G is *again reflected along* G H and is divided at H into two



portions, one H I leaving the glass and the other H K reflected within it. Thus there is a series of rays B C, E F, H I, L M, .. of which the second E F is the brightest, the subsequent ones H I, L M, .. rapidly diminishing in brightness, and each of them causing an image of the luminous object A. In the ordinary use of the looking glass even the image belonging to B C is not noticed without a little attention, but if we take the obliquely reflected light of a flame we readily perceive several of the images.

On this account the silvered mirror can hardly be used where great precision is wanted; in the few cases where it can be used, the utmost care must be taken to have the two surfaces parallel to each other, so as to ensure the parallelism of B C and E F. When the glass is not equally thick throughout, the lines B C, E F, H I, &c., are inclined, producing great separation in the images, and consequent confusion.

LESSON XXXVII.

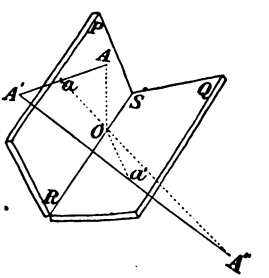
ON THE REFLEXION OF LIGHT BY TWO FLAT SURFACES.

It was stated, in the preceding lesson, that the plane passing along the incident and the reflected rays passes also along the normal to the reflecting surface; and that the angles of reflexion and of incidence are alike. No reason, however, was mentioned for our belief in the truth of these statements. It is quite easy to make observations sufficient to convince us of their truth, when the reflexion is at the outer surface of a solid body, and as yet no instance is

known in which they fail; but when we examine the motion of light in the interior of transparent substances, we meet with cases in which the angle of reflexion is neither equal to the angle of incidence nor in the same plane with it; this may show how very dangerous it is for us to generalize the results of our limited observations.

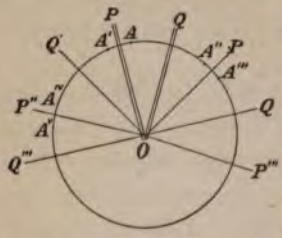
Light which has been reflected from a polished surface seems to have been changed in direction only; modern researches, however, have shown that light acquires certain new characters on being reflected from some surfaces. As far as observations yet go, the angle of reflexion from the outer surface of a solid or fluid body is equal to that of incidence, whether the incident light have or have not been previously reflected.

If a luminous point A be placed in front of a mirror P, the light from it, after being reflected from the mirror, proceeds as if it had come from the point A′. If this reflected light be again reflected from a second mirror Q, it will proceed as if it had come from the point A″ placed as far on the other side of the mirror Q as A′ is in front of it.



Every plane passing along A A′ is perpendicular to P, and every plane passing along A′ A″ is perpendicular to Q, wherefore the plane passing through the three points A, A′, A″, is perpendicular to both of the reflecting surfaces. Let these surfaces be extended to meet along the line R S, that line must be normal to the plane of A A′ A″, and if O be its point of *intersection, the lines* O A, O A′, O A″, O α, O α′ are all

perpendicular to R S. Also, since a A′ is equal to a A, O A′ is equal to O A, and, in the same way to O A″, wherefore the luminous point A and its images lie in a plane perpendicular to the common section of the mirrors, and in the circumference of a circle having its centre at O. Let us suppose the planes of the two mirrors to be perpendicular to the plane of the paper along the two lines O P, O Q, and A to be the position of the luminous point, then all the images of A will be in the circumference of the circle described with O A as a radius. On making the angles Q O P′, P O Q′ each equal to the inclination of the mirrors, we obtain O P′ the image of O P as seen in O Q and O Q′ the image of O Q as seen in the mirror O P. These images, again serving as objects, have their images at O P″ and O Q″ which are the images of the mirrors as seen by two reflexions. In the same way continuing the series of equal angles on each side we get O P‴ and O Q‴ the images of the mirrors as seen by three reflexions.



The images of the luminous object A are placed symmetrically in these angles alternately left and right; they become fainter at each successive reflexion, and, if visible far enough, the series meet and overlap each other.

When the angle P O Q of the mirrors is an aliquot part of a half-turn those images which overlap coincide exactly, so that any set of objects placed in the angle P O Q is reproduced alternately right and left all round the centre O. This is the principle of the amusing toy called the *kaleidoscope*.

Exercise 1.

The angle P O Q of two mirrors being one-tenth part of a turn or $36° = 40^c$, draw the images of a triangular object placed unsymmetrically between them, as formed by the successive reflexions.

Exercise 2.

A hollow prism is formed by the meeting of three mirrors at angles of 60°, and an irregular object is placed within the prism; draw the positions of the object as seen by repeated reflexions.

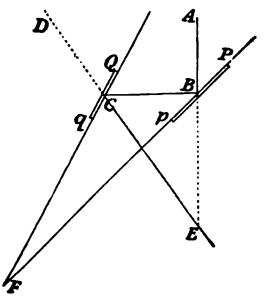
Exercise 3.

A hollow prism is formed by the meeting of three mirrors at inclinations of 45°, 90° and 45°; draw the appearance of the images of an irregular object placed between them.

LESSON XXXVIII.

ON HADLEY'S QUADRANT.

When the incident ray is in a plane perpendicular to each of two mirrors, the direction of the light, however often it may be reflected, is confined to that plane. Let then A B be the direction in which light coming from a remote object A falls upon the mirror P *p*, supposed to be perpendicular to the plane of the paper. On making the angle P B C equal to P B A *we obtain B C the* direction of the reflected light. If this



light meet the surface of a second mirror Q q, also supposed to be perpendicular to the plane of the paper, at C, its path C E, after the second reflexion, is found by making q C E equal to Q C B, and the eye at E receiving it, will see as if the object A were far off in the direction E C D. The first reflexion gives the twin or symmetric of the object, the second reflexion gives the twin of that twin, that is the *equal* to the original object, and thus, as seen in the second mirror Q q, things appear not reversed but in their natural positions.

If we continue the line A B to meet D C in E, the angle A E D is the change of direction, or the difference between the *real* direction E B A of a remote object as the sun or a star, and the *apparent* direction thereof as seen in the second mirror; and if the faces of the mirrors be extended to meet in a line normal to the paper at F, the angle Q F P measures their inclination. Now A E D is the difference between D C B and C B E, while Q F P is the difference between Q C B and C B F; but D C B is the double of Q C B and C B E is the double of C B F, wherefore A E D, the change of direction, is double of Q F P, the inclination of the mirrors.

This theorem "that the change of direction of twice reflected light is double of the inclination of the two mirrors" is only true when both the incidences are in one plane. If the mirror Q q were not perpendicular to the plane of the paper, that is to the plane of the first incidence, the reflected ray C E would not be in the plane of the paper, and could not meet the continuation of A B; neither would the angle Q F P measure the inclination of the reflecting surfaces.

The earlier astronomers measured the angular distance *between two stars*, or between the moon and a star, by

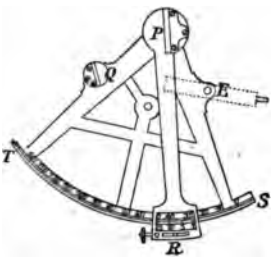
help of a jointed rule, having a sight-hole at the joint and a vizé at the end of each of the arms. They directed the one arm to one of the stars and opened out the instrument until the other arm pointed to the second star. The performance of this operation is by no means easy when the objects are fixed; it becomes more difficult when, as in the case of the stars, they are moving; and is altogether unmanageable when the observer is on board ship. But the navigator needs to measure the moon's distance from the stars, for the purpose of finding his longitude; and the sun's altitude above the horizon, in order to get his latitude. It was therefore a very great want, the want of some ready means of measuring angles at sea.

The difficulty in using the double-sighted instrument is, that we cannot look to both objects at once. Now by placing the eye so as to look partly past the upper edge of the mirror $Q\,q$ and partly into the mirror, we shall receive light from some remote object in the line E C D and, at the same time, reflected light from the object A, so that the eye will see both objects together. If the two mirrors be both fixed to one frame their angle will not change although the instrument be not held steadily, and the apparent coincidence of the objects will not be affected by the unsteadiness. Hence if such a frame be furnished with the means of changing the inclination of the mirrors to suit the angular distance of the objects, and also with the means of measuring that inclination, we shall have an instrument fit for measuring angles at sea.

Such was the instrument presented to the Royal Society of London on 13 May, 1731, by John Hadley, Vice-President of the Society. It was arranged for measuring angles up to 90°, and is therefore called Hadley's Quadrant, but *the principle of its action* is not thus limited. It is remark-

able that in Thomas Sprat's 'History of the Royal Society,' published in 1667, mention is made of "A new *Instrument* for taking Angles by reflection; by which means the Eye at the same time sees the two Objects, both as touching in the same point, though distant almost to a semicircle: which is of great use for making exact *observations* at Sea;" so slowly are obvious and even needful improvements introduced.

The instrument as usually constructed for measuring angles up to 120° is shown in the figure. It is composed



of a frame occupying rather more than the sixth part of a circle, on which account it is called the *sextant*. The mirror P is fixed on a plate which turns on the centre of the circle and has an arm extended to the circumference. The second mirror Q is fixed on a small plate capable of being adjusted; and a stand E is fixed for holding the eye-hole, or for carrying a small telescope pointing to Q.

The mirrors P and Q are commonly made of silvered glass, in which case the greatest care must be taken to have the glass of equal thickness, lest there be double images. When they are of speculum metal, the images are much more clearly defined than with silvered glass, and the coincidences much more readily examined, but there is this inconvenience, that the surfaces are easily tarnished.

The mirror Q is made of a height considerably greater than that to which the telescope reaches, but it is silvered only to the height of the centre of the telescope, so that the light coming directly from the remote object D through the clear part of the glass, may enter the telescope. Lastly

the edge of the circle is divided into half-degrees which are numbered as whole degrees, because half a degree of inclination of the mirrors gives one degree of angle between the two objects; and the end of the arm has marks on it at R for reading the divisions on the circle or *limb* as it is called. When the movable mirror P is parallel to the fixed mirror Q, the *reader* at R should show 0 degrees, in which case the number of degrees as shown by the reader when in any other position will be twice the inclination of the mirrors.

LESSON XXXIX.

ON THE ADJUSTMENTS OF HADLEY'S SEXTANT.

THE angle between the two remote objects A and D is double of the inclination of the two mirrors only when all the incidences and reflexions are in one plane. Hence each of the mirrors must be perpendicular to the plane of the instrument, and the telescope or the line of vision E C must be parallel thereto.

The centre of the eye-hole at E and the upper edge of the silvered part of the mirror Q should, therefore, be at the same height above the plate; moreover the eye-hole should be as large as the pupil of the eye, in order to permit of the comfortable use of the instrument, for the essential condition of the performance is that the pupil be so placed in relation to the edge of the mirror as that it shall receive partly light from above and partly light from below that edge.

When the telescope is used, its axis must be placed parallel to the plate, but it is not necessary that its axis *be at the height of* the upper edge of the mirror Q; the

coincidence of the two objects should, however, always be observed in the middle of the field of view.

In order to allow the mirror P to be made exactly perpendicular to the plane of the instrument, it is fixed to a kneed piece with a triangular sole, the under surface of which is slightly rounded. This sole is bound down to the plate by means of three small screws, so that, by slackening one and tightening another of these, the mirror may be inclined outwards or inwards. The exact perpendicularity is judged of by directing the mirror towards the middle of the arc S T, and examining whether the image of the part T, seen by light reflected from the mirror, agree in height with the corresponding part of the limb towards S. The screws are tightened or slackened until an agreement take place.

When the reader or *index* R is at the zero of the scale, the mirror Q should be parallel to P. For the purpose of examining this parallelism we look through the telescope at some very remote object, so remote that the instrument, if placed there, could not be perceived. If the mirrors be not quite parallel, two images of the object will be seen, one by direct light, the other by light reflected from the mirror P to Q, and thence by Q to the eye. We have to adjust the mirror Q so as to make the two images agree.

In order to admit of this adjustment, the knee carrying the mirror Q is supported on a circular plate capable of being turned partially round, and often having a short arm and adjusting screw. A small cavity is prepared in this plate for receiving a short stiff spiral spring on which the sole of the knee rests, and the knee is kept down on this spring by a finger-screw, thus enabling us to regulate the perpendicularity of the mirror. Holding the instrument *horizontally and* looking, say, at a distant spire let us

suppose the images to be displaced as in the figure; we should, by means of the finger-screw, bring both to the same level, thereby setting the mirror Q perpendicular to the plane of the instrument. Then, by means of the horizontal adjusting screw, we should bring the two images quite to agree. In some instruments, perhaps the better ones, the horizontal adjusting screw is wanting; the exact agreement of the images is, in that case, to be made by the motion of the arm P R, and, as the index no longer points to zero, its reading is carefully noted and recorded as the *index error*. With a good instrument, well cared for, the index error may remain unchanged for a long while. Should, for example, the index, instead of showing zero, show 2′ 20″, we shall have to subtract 2′ 20″ from every reading, and the *index correction* is then − 2′ 20″. But if the index should happen to be on the opposite side of the zero, say at 359° 58′ 30″, the instrumental correction would be additive or + 1′ 30″.

When the sextant has thus been adjusted we are ready to measure the angular distance between two remote objects. Looking steadily at the left-hand object through the telescope or through the clear part of the mirror Q, we bring the plane of the sextant to pass through the other object also, and slowly push the index R along the limb till the right-hand object come into view. This operation requires some practice, particularly when we have to do with two stars, because it is somewhat difficult to judge of the slope. In measuring the moon's distance from a star we find it much easier to bring the moon's reflected image to the star, than to bring that of the star to the moon; wherefore if the moon be the left-hand object we prefer *turning the sextant* upside down for the purpose of bringing

both into view at once. When we have succeeded in getting a sight of both objects at once, there is no further difficulty. When a seaman is taking a lunar distance, he gets that distance approximately from the almanac, and can set his index to the reading, sufficiently well to enable him to see the moon and star at once.

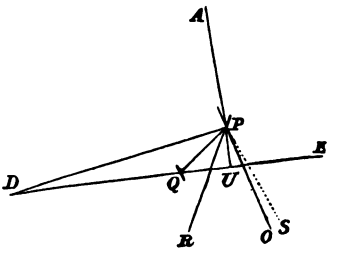
LESSON XL.

ON THE PARALLAX OF HADLEY'S SEXTANT.

We have hitherto only spoken of the sextant as useful for measuring the angles between remote objects, objects so remote that the instrument itself may be regarded as a mere point in comparison. Let us now see how it may be used for near objects. Returning to the figure of Lesson XXXVIII., it is obvious that the vertex of the angle which we measure is at E, and that the position of the point E must vary with the inclination of the mirrors, being, however, always confined to the straight line D C E. When the objects are remote, this changeability of the vertex causes no inconvenience; but for near objects we must have a fixed vertex, which naturally must be the centre of motion of the arm P R; that is the centre of the graduated arc S T.

Let the objects be close at hand, at A and D as in the present figure; we wish to measure the angle A P D, and for that purpose bring the mirror P into the position P R equally inclined to the lines Q P, P A so as to cause the image of A and the object D to appear together. If the instrument be in adjustment, the zero is in the *direction P O* parallel to the mirror Q, and the angle

O P R is read on the limb, giving the inclination of the line P A to E D. If now we bring the mirror P into the direction P S equally inclined to D P and P Q, the direct and the reflected images of D will appear to agree, the angle S P R is then half of D P A. Wherefore if we take the *index correction* by help of the object D, that is the object seen directly, and apply that correction to the reading, we shall have the angle as measured at P the centre of the instrument; the angle S P O is evidently half of P D E.



The angle P D E which measures the difference between the directions of the object D as seen from E and as seen from P, is called its *parallax*, from the Greek word παράλλαξις signifying *error*, *change* or *difference*; it increases as the object comes nearer. If we draw P U perpendicular to the line of vision E Q, we have D P: P U :: R : sin P D U, whence the parallax for a given distance may be computed; thus P U being $1\frac{1}{2}$ inch and P D 100 feet, the tabular sine of the parallax must be $\frac{1}{800} = \cdot 001\,25$, which is that of 4′ 18″.

Exercise 1.

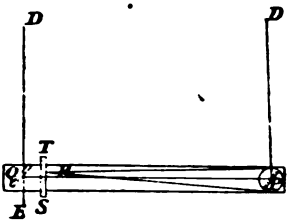
The distance P U being 2·4 inches, what is the parallax of an object three miles off?

Exercise 2.

At what distance must an object be placed to have a *parallax of one minute*?

Digression.

Seeing that we may compute the parallax when the distance is known, it follows that, by observing the parallax, we may determine the distance of the object; only the parallax in a sextant is so small that any moderately great distance could not be obtained with precision. In order to augment the accuracy of this kind of observation we may separate the mirrors P and Q to the distance of several feet. Placing Q at one end of a board 6 or 10 feet long, and setting the mirror P at the other end on a movable plate provided with a long index arm P R for reading the graduations on the arc S T, we have the instrument arranged for measuring parallax. In order to find the zero point we must use an exceedingly remote object as the moon when rising or setting; and in order to augment the accuracy we may examine the coincidences by help of a telescope.



Exercise 3.

Where P Q is 5 feet, what is the parallax of an object at the distance of 1000 yards?

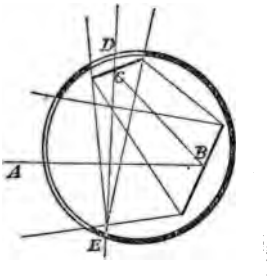
Exercise 4.

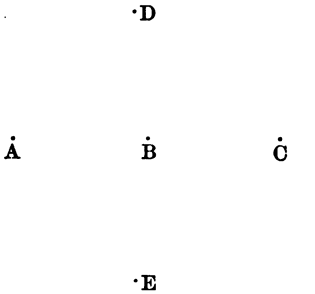
Make a list of the parallaxes for 50, 100, 150, 200, 300 . . . . 900 yards.

LESSON XLI.

ON THE OPTICAL SET-SQUARE.

In carrying on measurements in the field we often have occasion to mark right angles, thus we may have to let fall a perpendicular from a point to a straight line, or we may need to stake out the corners of a rectangular inclosure. For this purpose we use a small instrument called the *optical square*, or more properly the *optical set-square*, which consists of two mirrors secured at an inclination of half a right angle, and inclosed in a cylindric box which need not be larger than the accompanying drawing.





In testing the accuracy of this instrument we proceed in the same way as for the common set-square. Having set three staves A, B, C in a straight line we place the optical square on the top of the intermediate staff B, and, placing it in the position shown in the previous figure, cause an assistant to place a fourth staff D to agree with the image of A as seen by reflexion; then turning the optical square over face for face we examine whether the same staff D now agree with the image of C. If it do *not, we cause a fifth* staff D′ to be placed; halve the dis-

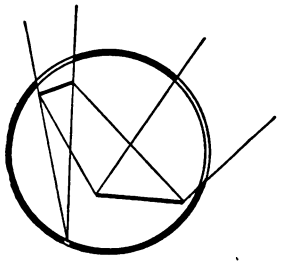
tance D D′ and place the staff D at the middle point. Having now obtained accurately the right angle A B D we adjust the inclination of the mirrors to suit.

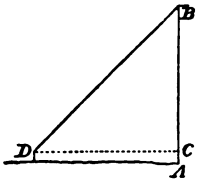
Should we fear that the line A B C is not quite straight we may repeat the trials by help of a staff E on the other side of the line A B C.

The box is made double, and by turning the outer cover partly round, the several openings are closed to protect the mirrors when not in use.

This instrument may be cheaply made up by cutting a piece of hard-wood with two faces inclined at 45°, and by binding two small pieces of looking-glass to those faces; taking care to select glass which does not give two images.

In the same way arrangements may be made for any other fixed angle. The half-square is useful on the drawing board, so also is a well-adjusted angle of 45°, in the field. The adjoining figure shows how the mirrors may be placed in order to give this angle. The height





of a building as A B may be neatly measured by help of this instrument. We fix the ring of the measuring *tape to the wall at* C on the level of the eye, and walk

backwards until the image of the top B be brought down exactly to C; the length C D of the tape together with A C gives the height A B.

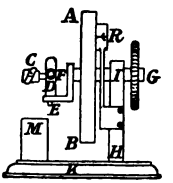
The student may readily construct this instrument by binding two pieces of mirror to two faces cut to be inclined 22° 30′.

LESSON XLII.

ON WOLLASTON'S REFLECTING GONIOMETER.

In measuring the inclination of two faces of a solid we may use the goniometer figured in Lesson VII. of Part II., provided the faces be sufficiently extensive. But it is often desirable to measure the edges of exceedingly small crystals, to which we should find it impossible to apply this instrument. The faces of small crystals are generally brilliant and flat, giving good images by reflected light; this circumstance led Dr. William Hyde Wollaston to contrive an instrument for measuring these edges by the comparison of the images.

The minute crystal C is cemented to a small stud in line with the axis F G of the graduated wheel A B; the particular edge to be measured being set, approximately by the eye, parallel to F G. This stud is fixed on a wire whose end is shown at D and which turns stiffly in its two sockets, so as to admit of adjustment in one direction; the frame carrying this wire, turns stiffly upon a joint at E, thus admitting of adjustment in a direction *at right angles to* the former, and in this way the axis



of the edge may be set truly parallel to that of the instrument, in which case each of the faces is parallel to that axis.

The method first put in practice for using this goniometer was to select two well-defined objects, say the upper and lower bars of a window; then, holding the instrument firmly on the table, to bring the image of the upper bar as seen by reflexion from the one surface to agree with the lower bar seen directly and to note the indication of the reader R. Then to turn the circle round until the image seen by reflexion in the second surface agree with the same bar, and to read again; the difference between the two readings is the supplement of the inclination of the two faces. Here the instrument must be held firmly on the table, and its axis, at each measurement, must be placed parallel to the lines which are compared.

In order to avoid the uncertainty and trouble of these repeated adjustments I place a mirror obliquely, on the stand K; the back of it is shown at M. This mirror is made, once for all, parallel to the axis F G, and the image of any remote object as seen in the face of the crystal is made to agree with the image of the same object as seen in the mirror; this coincidence shows the two reflecting surfaces to be parallel. Reading the indication, making the second coincidence and again reading, we get the inclination of the two surfaces of the crystal.

The mirror M should not be made of a brightly reflecting substance; a piece of obsidian or of black-glass answers better, because too bright an image would overpower that given by almost any crystallized substance. In order to test its adjustment, we compare the image in a crystal with that in the mirror looking down past the crystal; *then, turning* the circle half round, we examine the co-

incidence of the image seen by reflexion first from the mirror and then from the crystal, with the object directly. Any lateral displacement is a proof that the plane of the mirror is not parallel to the axis of the instrument.

It is easy to use the reflecting goniometer when the crystal is so small as only to cover part of the pupil of the eye; because then the direct light passing the sides of the crystal enter the eye.

Any turning lathe may be used as a reflecting goniometer, by graduating the pulley and fitting a piece analogous to D E F on the point of the spindle.

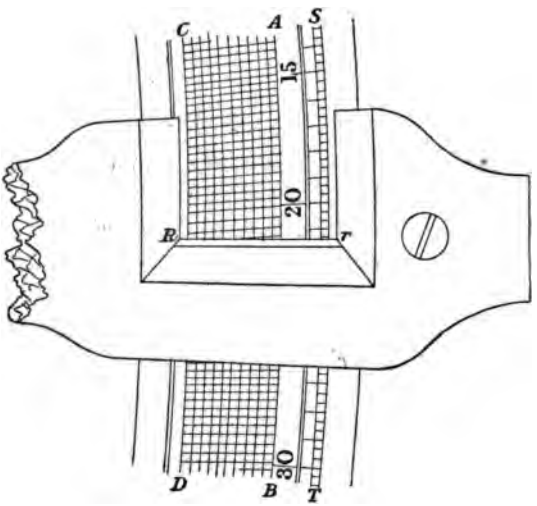
LESSON XLIII.

ON THE VERNIER.

An ordinarily good eye is able to note the coincidence of the two images in the sextant to within half a minute of a degree, and therefore, in order to get the full advantage of Hadley's contrivance, we must have the divisions on the limb made to this degree of nicety. Now if the radius of the sextant be one foot, the circumference of the circle is about 72 inches; this has to be divided into 720 parts each marked on the limb as one degree; so that one degree of angle must correspond with about one tenth part of an inch, and one minute of angle with one six hundredth part. To read to half a minute of the ancient, or to one minute of the modern division, we thus require graduations to about the thousandth part of an inch, on an instrument of this size. We must then have contrivances for making and for reading divisions to this *nicety*. *The making* of the divisions will be considered

under the heads of Graduation and Workmanship; for the present it is enough to treat of the mode of reading.

When using a simple pointed index, we have to estimate its position between two of the marks on the limb, and this estimate is liable to uncertainty. The first contrivance used for subdividing the spaces on the limb, was a modification of the method proposed in 1573 by Thomas Digges of Barham in Kent, for the graduation of linear scales. The accompanying drawing is a copy, full-size, from the limb of a beautifully-made mahogany quadrant of 16 inches radius, bearing the inscription I. URINGS. Fecit. LONDON. The divisions are on boxwood.



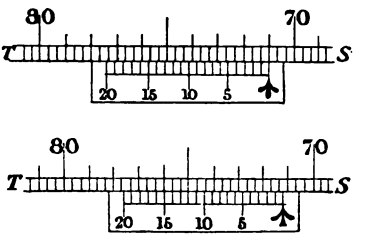
Each degree is divided, on the line S T, into three parts of 20′ each, and the reader or index is the edge R *r* of an inlaid piece of ivory. The reading, as shown in the figure, is somewhat more than 21° 20′.

ie divisions on S T are transferred to the two con-

centric lines A B, C D and each point in A B is joined with the point one step in advance on C D; the interval between A B and C D is divided by nine concentric lines into ten equal parts, so that each succeeding crossing on one of the oblique lines is in advance by the tenth part of 20′, that is by 2′. Now R *r*, as shown, crosses the oblique line drawn from 21° 20′ at the fourth division, wherefore the index is pointing to 21° 28′.

Although it be a very great improvement on the simple index, this contrivance fails to satisfy our requirements; on account of the breadth of any line visible to us, it is difficult to judge of the oblique intersection; this plan is now superseded by the invention of Pierre Vernier, a gentleman of Franche Comté.

If the edge S T of the limb be marked in distances of 20′ each, and if, on an adjoining movable piece, called the *Vernier*, there be made divisions of 19′ each, 20 divisions on the vernier will be as long as 19 divisions on the limb. Let the beginning or 0 of the vernier be brought exactly opposite to any one of the marks on the limb, then the mark 1 on the vernier must be behind the next mark on the limb by 1′, the second mark is behind by 2′, the fifteenth mark is behind by 15′ and so on. Hence if we now shift the vernier along until its thirteenth division agree with the next division on the limb we shall have brought it forward by just 13′, and the reading, as shown in the second figure, is 71° 13′.



If the division on the vernier be made one minute more

than that on the limb, an analogous result is obtained but with the disadvantage that the numbering must be done in the opposite direction.

In the above figures, a distance of nineteen spaces on the limb, is marked on the vernier and there divided into twenty parts; the difference being a twentieth part of a division. The same method is applicable to any other number; thus if it be required to subdivide the graduations into 60 parts, 59 of these graduations are marked on the vernier and thereupon divided into 60, each vernier division being $\frac{59}{60}$ of the division on the limb and consequently $\frac{1}{60}$th of a division shorter.

EXERCISE 1.

Having graduated the edge of a scale into inches and tenth parts, make a vernier to read upon it hundredth parts of an inch.

EXERCISE 2.

The limb of an instrument is divided into degrees and half degrees: construct a vernier to read single minutes.

EXERCISE 3.

Each tenth of an inch being divided into five parts, arrange a vernier to read to thousandth parts of an inch.

EXERCISE 4.

The degree being divided on the limb into four parts, arrange a vernier to read to quarters of a minute.

EXERCISE 5.

When the degree is divided into six parts, show the verniers for reading respectively to minutes and to sixth *parts of a minute.*

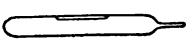
The eye is able to judge with great nicety of the opposition of two clearly defined lines of equal breadths; when these are or should be in continuation of each other, an error of the thousandth part of an inch is quite perceptible without the aid of a magnifying glass; but when the surfaces of the vernier and of the limb lie differently the agreement is not so easily judged of.

LESSON XLIV.

ON THE SPIRIT-LEVEL.

In order to determine completely the direction of a line, we must ascertain the angles which it makes with at least two lines of known directions. The direction of the plumb-line or of *gravitation* is necessarily taken as one of these, and we refer all our field measurements to the *horizontal plane* which has this direction for its normal. The surface of any fluid at rest, except a small portion round the edge of the containing vessel, is horizontal, and if we suspend a plummet above it, we find that the image of the cord, as seen by reflexion in the fluid-surface, is in exact continuation of the line of the cord itself.

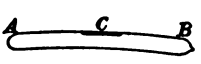
The most convenient and by far the most accurate instrument for indicating horizontality is the *Spirit-Level*, the invention of which is ascribed to Robert Hooke. A glass tube, sealed at the one end and drawn out to a very narrow neck at the other end, is filled with some fluid,



leaving enough of empty space to form a short air-bubble when the tube is held horizontally; the tube being then held upright has the narrow neck melted off in the flame

of a blow-pipe, so as permanently to shut in the fluid and prevent change.

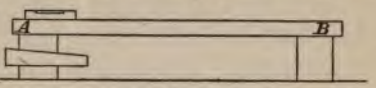
When this instrument is held slightly inclined, the air-bubble keeps to the higher end. If the inside of the tube were perfectly straight, the slightest inclination the other way would cause the bubble to move along the whole length of the tube, we should be able to tell which end of the tube is the higher, but we should not be able to estimate the degree of inclination. Instead of a straight tube we therefore use one slightly bent, in which case the air-bubble rests at C the uppermost part of the tube. Supposing that A B is a circular arc, and that its length is marked off in degrees of the circle, a change of inclination amounting to one degree would be accompanied by a motion of the bubble through one division, and thus we should be able to tell the inclination accurately within the range of the instrument.



It is not easy to get a piece of glass tube quite straight. The glass-blower forms what may be described as a thick-sided bottle on the end of his iron tube; while this is still hot and soft, an assistant sticks another piece of hot glass to the end of the bottle and the two men walk backwards so as to draw the bottle out into a long narrow tube. The weight of this necessarily bends it, and so renders it fit for the more ordinary purposes of levelling. Pieces free from internal knobs and irregularities are chosen and sorted according to the curvature.

Alcohol, not being liable to freeze, and having very little viscidity, is almost the only fluid used in filling these tubes. Chloroform, being much heavier and at the same time very mobile, is also coming into use, but as it boils at a moderate heat it can hardly be used in hot climates.

All such levels should have divisions either engraved on the glass or marked on the setting; and the user should always test the value of the divisions, in order that he may have confidence in using the instrument. The mason or carpenter may place his spirit-level on a batten A B as long as



he can procure, supporting the one end by a wedge. By pushing in or drawing out the wedge he can bring the air-bubble to the different divisions on the scale, and by comparing the change in height of the end of the beam with the distance between the two supports, he may readily compute the value of the scale, as, for example, in how many inches horizontal there is a rise of one inch for one division of the scale. The knowledge of this will save him a great deal of trouble in his work, for by reading the error as shown on his level scale, he can calculate how much is to be pared off or put on.

Exercise 1.

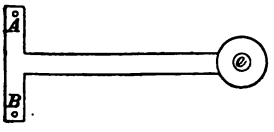
In order to get the radius of curvature of a spirit-level, a temporary scale of inches and tenths was laid along the edge of the glass. The instrument was then laid on a batten supported on props 9 feet 7 inches apart. These props were adjusted, by means of a wedge, until the air-bubble was close to one end of its scale and the height of the changeable prop noted. This height was then altered until the air-bubble was brought near to the other end of its scale; when it was found that a change of ·32 in the height of the prop had caused the bubble to move 1·37. The radius of curvature is then found from the proportion ·32 : 1·37 :: 115 : 492, so that the radius of curvature has been 492 inches or 41 feet.

If we wish to graduate this level to show angles in minutes, we compute the length of an arc of one minute on this circle: the result is ·143 or almost exactly $\frac{1}{7}$ of an inch. Or if we wish to show the slope in parts of the distance we observe that a motion of ·492 (rather less than half an inch) corresponds to the thousandth part of the radius; wherefore a scale of divisions each ·0492 (rather less than the twentieth part of an inch) would show the inclination in ten-thousandth parts of the distance.

Operating in such a way as this, the student should graduate his spirit-level, or test the graduations if these be already made.

Levels which are to be used in the construction of surveying or astronomical instruments are made with greater care, the insides being ground on brass rods almost cylindrical, but slightly raised in the middle. In this way the interior of the tube, if ground all round, would be barrel-shaped. Though, strictly speaking, such a form be not self-gliding, the deviation is so small that no inconvenience is experienced in the actual manufacture.

For the purpose of graduating delicate levels we use a long bar having a short cross-head A B at one end; this

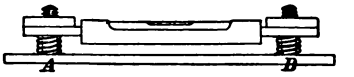


cross-head rests on two steel pins at A and B. The farther end *e* of the bar is supported on a screw having a graduated wheel; this screw serves the purpose of the wedge in the arrangement previously described; there is no difference in the principle, only, since the pitch of the screw may be accurately measured, we are able to get much greater precision in the results.

Exercise 2.

The screw *e* has 50 turns to the inch, and the head of it has 60 divisions: what must be the distance from *e* to the middle of A B, in order that one turn of the screw may give an inclination of one minute, and, of course, that one division of the screw-head, may give one second of inclination?

A convenient arrangement is shown in the adjoining figure. Projecting pieces from the ends of the level-case rest on two spiral springs and are bound down to the sole-plate A B by the screws with graduated nuts. These enable us to make the middle of the internal upper surface of the glass parallel to the under side of the sole-plate. For this, we place

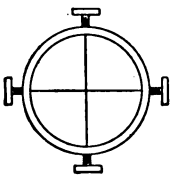


a try-plate or other flat surface as nearly horizontal as we can judge and place A B upon it; by means of the adjusting nuts we bring the air-bubble to the middle of the scale, read and note down the indications on each nut. We then reverse A B end for end; if the air-bubble be not at the middle we bring it there by moving one of the adjusting nuts; read the indication, take half the sum of the readings in the two positions and set the nut to the mean thus obtained. The level is now almost in adjustment; we use it to set the try-plate horizontally, and repeat the verification. When the try-plate is truly horizontal and the level in adjustment, the air-bubble remains at the middle in whatever direction the trial may be made.

LESSON XLV.

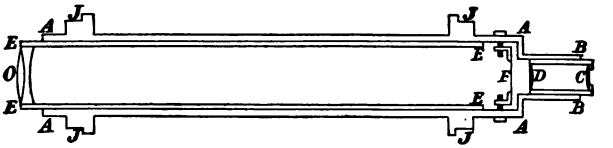
ON THE TELESCOPE SIGHT.

An essential part of all instruments for observing direction is the *pointer* or *sight*, called by the older writers alhidad from the Arabic هداة and هدايۃ, *hidad* and *hidaïet*, direction. When the telescope is to be used as a sight, fibres are stretched in the focal plane; they are fixed on a piece called the *field-bar*, kept in its place by four adjusting screws. Any wire that can be drawn is far too thick for the purpose; spider webs are generally used. Those caterpillars which feed on the leaves of trees and which afterwards bury themselves in the ground to pass into the state of chrysalis, let themselves down by a fine thread. These threads answer better than the spider line. Dr. Wollaston proposed to draw platinum wire thin enough for this purpose, by casting a cylindric silver ingot round a platinum wire, drawing both together and removing the silver by means of nitric acid: he succeeded in making platinum wire so fine as the ten-thousandth of an inch in thickness. The writer of the present article, having obtained some of this wire, attempted to place it across the field-bar of a theodolite, but quite failed. The reason of the failure is very simple. The wire is incapable of being stretched without rupture, and the strain sufficient to tear it is quite imperceptible, hence it could not be held even between the points of two hair pencils or be brought in contact with the field-bar without giving way. Whereas the spider line is elastic and may be handled with great freedom.



The surveyor should be able to replace the fibres in his telescope; he need not be alarmed at any difficulty in the matter. The cuts in the surface of the field-bar must be cleaned out with a little alcohol, or with a drop of the solution of caustic potass on the point of a soft stick. The spider line is most conveniently held between the legs of a pair of compasses; it is brought down into one of the cuts, moved slightly lengthwise to make sure of its getting to the bottom, and is to be fastened by a minute drop of spirit lacquer or varnish from the point of a needle. The stray ends must be carefully removed.

The field-bar F is secured to the principal tube A A A A of the telescope by its four adjusting screws; and, in order



to view the wires, the eye-piece C D is provided with a sliding tube to permit of its being adjusted to distinct vision. This adjustment depends on the peculiarity of the eye of the observer, and is the first to be made when we are about to use the instrument.

The object-glass O is set in the farther end of a tube E E E E which slides within the principal tube, and is commonly provided with a rack and pinion for giving it longitudinal motion. In this way we are enabled to bring the conjugate focus of the object at which we are looking to agree with the plane of the wires. If, on moving the eye from side to side of the aperture at C, the image appear to shift upon the wire, the adjustment to focus is not true; when that image seems to move in the same

direction with the eye, it is beyond the plane of the wires, and the object-glass must be brought nearer.

We shall afterwards have to examine carefully the construction of the object-glass O, and the arrangement of the eye-lenses C and D; it is enough to remark now that the object-glass consists of two lenses, the outer one of crown glass and convex, the inner one of flint glass and concave, designed to correct, as well as may be, the actions of lights of different colours, and also to lessen the spherical aberration. The lens D, called the field-lens, is placed so as to form an image of the object-glass at a distance a little beyond the eye-lens C, thus enabling us to bring the eye very close to that lens and so to obtain a large field of view.

Thus much for the optical performance of the instrument; we have now to make sure that the direction in which we seem to be looking, agree with that of the axis of the tube, or with some other line fixed in regard to it. The construction of a telescope for levelling affords perhaps the best illustration of the principle.

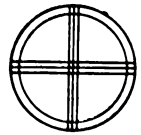
In order to obtain a mechanical definition of the axis, two strong rings are soldered to the principal tube and are turned concentric with it, so as to form two journals J, J of exactly equal diameters, and each with a ledging to prevent the tube from slipping off from its supports. The supports are pieces, called V's or Y's, with angular notches, into which the journals drop. If circular bushes were used there would be the danger of any small particle getting below at the middle and causing uncertainty; there would also be the necessity for accurate lining. When we come to examine the

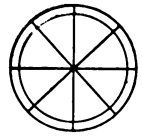
doctrine of *maximum* and *minimum* (greatest and least) we shall find that the *best* angle of slope is half a right angle.

If the telescope be laid in its V's, if the crossing of the wires be brought to agree with the image of some point in a distant object, and if the tube be then turned round on its axis, the crossing will only remain at the same point of the object when the line of vision is parallel to the axis. When turned half round, the crossing is as far on the one side of the axis as it had been on the other side; wherefore we halve the difference, if there be any, by means of the adjusting screws of the field-bar, and continue the trials and adjustment until no motion be perceptible. We are then sure that the optical axis of the telescope agrees with its mechanical axis.

The tubes of telescopes are made by drawing: a sheet of brass is bent into the form of a cylinder, and its edges are joined by solder; a steel core being inserted, this is then drawn through a hole in a steel plate, is compressed in diameter and somewhat lengthened; the operation being repeated until the desired size be reached. Now the soldered side draws differently from the rest, so that hardly a tube is quite straight, almost all being curved toward the seam. This being the case, it follows that the optical axis may change with the protrusion of the inner tube, and that, though adjusted for one distance, the telescope may not be in adjustment for another distance. For this reason important measurements must be made first with the telescope in one position and then with it in the opposite; the one reading being in excess and the other in defect, half their sum is free from the error of lining.

The simple cross in the field-bar is enough for the ordinary work of the surveyor, but for many special purposes it is convenient to have one or more parallel lines on each side of the centre. The separation of these should be some definite quantity in minutes of a degree, or in fraction of the distance; it may be estimated by placing a divided scale at some considerable measured distance from the object-glass, by observing the number of intercepted divisions and thence computing the angle. But it is to be noticed that this computation gives the angle subtended by the separation as at the centre of the object-glass; allowance must therefore be made for the protrusion of the tube, in order to get the angular separation corresponding to the principal focal length of the glass. A scale of decimal parts of the focal length may be engraved on the sliding tube.





Thus in order to measure the angular distance of the wires in a surveying telescope, a scale was set up at the distance of 400 feet, and the interval between the wires was observed to be 8·5 inches. Now 1′ at that distance is almost exactly 1·4 inch, wherefore the angular separation of the wires is $6'\frac{1}{14}$, intended, probably, by the maker to have been 6′. But for this distance the tube has been lengthened; now the focal distance of the object-glass was 10 inches, wherefore the elongation is $\frac{1}{48}$ of an inch, (the fourth term of the proportion 4800 : 10 :: 10 : $\frac{1}{48}$). The wires subtend the angle of $\frac{85}{14}'$ at the distance $10\frac{1}{48}$; they must subtend a greater angle at the distance 10 inches, in the ratio of $10\frac{1}{48} : 10$. Hence to get the of the wires for distant objects, we have the $10 : 10\frac{1}{48} :: \frac{85}{14} : \frac{8177}{1344} = 6\frac{1}{12}$ very nearly. This

correction being, in the present example, too small to be perceptible.

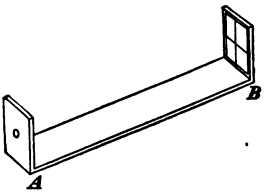
A field-bar having four wires crossed at equal angles is useful in making observations on the stars.

LESSON XLVI.

ON THE THEODOLITE, OR SURVEYING CIRCLE.

ASTRONOMY, practical surveying, and several branches of the arts, were introduced into Europe from the Arabs; hence the frequent occurrence of Arabic technical names. The earliest surveying instrument was what we call the *plane*-table, a board having the paper stretched on it, and fixed on a stand, which admitted of the board being placed horizontally.

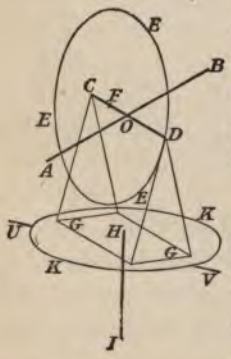
In order to observe the directions of the surrounding objects, a large straight rule was provided, having a piece set up at the one end A, with the sight-hole pierced in it; and, at the other end B, an open frame carrying two crossed wires. This was called by the Arabs *al hidat* (the hidat or pointer). The table being placed at, say, the meeting of two straight sides of a field, the angle of those sides was transferred to the paper by laying the pointer in the direction first of the one and then of the other, a line being drawn along the edge of the rule each time.



This very inconvenient process is now superseded by th *Theodolite* of Thomas Digges, a *new* edition of whose b

on 'Practical Geometry' is dated London, 1591. The name is not very appropriate to the instrument; it is a compound from the Greek θεαομαι to examine and δολιχος long or far, and would mean the *measurer of distance*, whereas the instrument only measures angles. Digges' first theodolite served only for horizontal angles; it consisted of a circle *of at least one foot in radius* "more if ye list," with the circumference divided into degrees and sixth parts, the degrees being numbered up to 360. A plain sight was fitted to turn on the centre; and the degrees indicated by it were to be noted diligently "upon some slate stone or table prepared." He afterwards removed the plain sight and placed instead an arrangement for angles in altitude measured by help of a plummet.

Arrangements for measuring angles of elevation were introduced gradually; even yet theodolites are manufactured which go only to 50° or 60° of elevation; these, besides being defective in their range, are incapable of self-verification. The complete theodolite must admit of its telescope being pointed in any direction whatever. It is then commonly called the *altitude* and *azimuth circle*, or the *surveying circle*. We proceed to examine minutely its construction.



The telescope, represented by the line A B, is fixed by its middle, O, to a horizontal axis C D, so that, when the axis is turned round, the telescope sweeps a vertical plane. A graduated circle E E is secured by its centre F to the same axis and serves to mark the direction of the telescope in altitude.

The horizontal axis C D rests on two pillars or supports springing from a plate G, which turns freely on a vertical axis H I; the pillars are so high as to permit the telescope to pass above the plate, and the continuation of the axis I H should pass through the point O called, therefore, the *centre of observation.* In this way the instrument has no parallax.

A graduated horizontal circle K is made concentric with the socket of the axis H I; and an index or reader attached to the plate G marks on it the azimets or horizontal directions; it is essential that this horizontal circle remain fixed while the observations are being made. Hence the first requisite is a steady support for the socket of the vertical axis.

Describing the *theodolitus*, Digges gives this direction: "In his backe prepare a vice or scrue to be fastened in the top of some staffe," and he so represents the instrument in his figures. Such a support was well enough for his purpose; it is, however, liable to flexure and to twisting and is quite unfit to meet the precision of modern instruments.

The method, still too commonly followed, of supporting the instrument by three legs jointed to a screw, was a decided improvement on the single stake, in so far that it secures definitely the position of the apex of the tripos; but it has the serious defect of depending on the resistance to flexure and tension afforded by the three legs. The stand introduced some forty years ago by Robinson, an eminent instrument maker in London, fulfils all the mechanical and geometrical conditions of stability, and should alone be used.

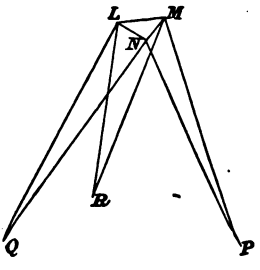
A rod held fast at the one end, resists pressure applied at the other end in three ways. It may be bent, twist-

or compressed; and the resistance to compression is so much greater, especially in a slender rod, than either of the others, that it alone should be trusted to. That is to say, in the language of Geometry, we must determine position by means of distances not of angles.

Now in order to fix the position of a solid body we must have data enough to determine separately the positions of three of its points, which must not be in one straight line. If held by only one point, the solid may be turned in any direction; if held by two points, it may rotate on the straight line joining these, as an axis; but when held by three points not in one straight line, it is fixed.

Again in order to determine the position of a point in space, using only distances, we must know three distances, as, for example, its distances from three known points not in one straight line; hence, in order to fix the position of a solid we require *nine* linear data.

Let us suppose, then, that L, M, N, are three points in the rigid frame which is to form the foundation of the theodolite. The *three* distances L M, M N, N L are already given by the rigidity of the frame, and there remain of the *nine* essential conditions, yet *six* lengths to be given, these must connect the frame with the ground.

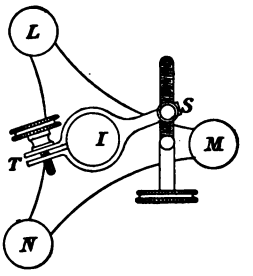


This analysis of the matter leads at once to Robinson's arrangement, which is too well known to need any detailed description. He connects each pair of the points L, M, N, with a single point in the ground; thus forming what may be described as three triangular legs L R M,

MPN, NQL, turning on long hinges LM, MN, NL. The superiority of this over the old tripod is so marked that no photographer would ever think of setting his camera on a common theodolite stand.

The socket for the axis of the horizontal circle must be adjustable in level; for this purpose it is cast in one with a three-armed plate, in each arm of which a levelling screw works; and the points of these screws rest on the three corners of the triangular frame LMN.

When the taper end of the axis I is placed in the socket, the horizontal circle K attached to the axis may be turned round and placed in the desired position: it has to be secured in that position. For this, a circular clamp or vice is prepared having a tail-piece S and a split end at T. This vice is elastic and when in its natural state is just free of the axis I; but when the split parts at T are brought together by help of a finger screw, the vice holds the axis firmly. The tail S is connected with one of the arms of the plate by a long screw U called a *tangent* or *adjusting screw*, by turning which the tail S may be pushed from or drawn towards the arm; this enables us to adjust the position of the horizontal circle with great nicety. The whole arrangement is called a *slow motion;* each of the other angular movements has its slow motion, acting just on the same principle, so that the description of this one may serve for all.



The axis of the circle K is pierced along its whole line with a conical hole to serve as a socket for the taper axis on which the plate G G turns. The axis of the horizontal

circle is thus a double cone, the outer and inner surfaces of which should be truly concentric.

The plate G carries a vernier for reading the divisions on the limb of the circle K; it is also furnished with a clamp and tangent screw.

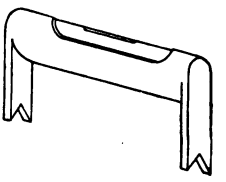
In order to make sure that the axis I H is truly vertical two spirit-levels are fixed on the plate G, each with its adjusting screws; the one of them placed parallel to the horizontal axis C D, the other at right angles thereto. These levels should be so delicate as to show easily, say by a motion of one-tenth of an inch, the smallest division to which the instrument is to read, but they should not be more delicate than this; it is convenient also that the graduation of their scales should correspond with those of the circles. Judging by the eye, we bring one of the levels to be parallel to the line joining two of the levelling screws, say to L M; and, by manipulating these screws, bring the air-bubble to the middle of the scale. Then, by means of the third levelling screw, we bring the air-bubble of the other level to its middle point. Turning now the instrument half round, we examine the levels, and if there be any error, we correct the one half by means of the levelling screws L, M, N; the other half by means of the adjusting screws of the levels; and revise the adjustment until there be no perceptible error. This adjustment should hold good whether the horizontal circle K be turned round carrying G along with it, or whether K be held fast and the plate G only turned round.

The journals of the horizontal axis C D should be of equal diameters, they rest in two V's formed in the tops of the supporting frames. This axis, carrying with it the telescope A B and the altitude circle E E, may be lifted out of the notches and reversed end for end, an operation

needed for verifying the perpendicularity of the telescope to the axis.

For the purpose of testing this we bring the cross-wires of the telescope to some well-defined distant object, securing both of the horizontal motions by their approximate clamps; then cautiously lifting the axis out of its V's, we turn it over, end for end, and examine whether the wires again come to the same object. As usual, we correct one half of any observed error by the screws of the field-bar, the other half by help of one of the horizontal tangent screws, and repeat the trials till there be no error.

It remains for us to examine the horizontality of the axis C D. This may be done by means of a *riding level*, having two notched legs projected downwards. These notches are placed on the journals of the horizontal axis, the length of the legs allowing the level tube to pass above the telescope and between the spokes of the vertical circle. The equality of the legs must be tested by reversion, and any inaccuracy corrected by help of a smooth file. Some instruments have one or both of the notches in the supporting frames at C and D movable and adjustable by screws; in others the adjustment has to be done by the file and smoothing stone. Another method of testing this horizontality will be mentioned shortly hereafter.



In order to read the divisions on the vertical circle E, a vernier must be supported from the frame at C, and there should be a corresponding vernier from the frame at the end D, or, at least, there must be the means for transposing the vernier. The vertical motion must have its clamp and tangent screw.

LESSON XLVII.

ON THE MODE OF USING THE THEODOLITE.

THE use of the theodolite is to compare the direction of some remote object with two known directions, one of which is that of gravitation, the other the *north line* or, if that be not known, the assumed azimet of some point of reference.

As an example, let A and B be two of the stations of a survey, marked by signal poles; and let the bearing of B as seen from A be known to be say 237° 43′. In order to observe the directions of other objects as seen from A, we remove the pole, plant the theodolite exactly over the place occupied by the pole, using a small plummet; and then make the principal axis of the instrument truly vertical by means of the levelling screws and spirit-levels. The next operation is to bring the vernier to read 237° 43′ on the horizontal limb, using the clamp and slow motion for the purpose. We then turn the whole instrument until the telescope be directed to the signal B, using the under clamp and tangent screw, and revising the reading to make sure that it has not been disturbed. The instrument is now ready for observations.

At this stage of our proceedings it becomes apparent that the accuracy of all the subsequent work must depend on the immobility of the lower part of the instrument. We have to unclamp the motion in altitude and also the upper horizontal motion, in order to turn the telescope to some new object. If only a moderate precision be needed, we may trust to the goodness of the workmanship for resistance to the strains thus occasioned; but, if we desire

acy, we must provide the means for detecting

any change of position, and for rectifying it when detected. The spirit-levels show any change as to horizontality, so that there remains the change in azimet.

On the under side of the horizontal circle K K, arrangements are made for securing a second telescope or *settler* U V. The settler being brought to point to some well-defined object, we secure it to the circle, taking care that the upper telescope points to the reference signal B, and that the vernier mark the 237° 43′ prescribed. If now the instrument should be displaced, the change must be shown either by the spirit-levels or by the settler, perhaps by both; and we are able to correct it, first by help of the levelling screws and then by means of the under tangent screw. At every subsequent operation, we must be careful to examine the collimation of the lower telescope.

Having directed the telescope to some new object or signal C, we read its azimet on the horizontal limb; and the indication on the vertical limb gives the direction of C in altitude.

The securing of the vertical circle E E upon its axis, and the placing of its reader, have now to be attended to. The earlier theodolites were arranged exclusively for surveying; the inclination being seldom great, it was natural to count from the horizon upwards and downwards. In this, however, there is considerable inconvenience; thus when we write say 13° 28′ we must mark whether it be *up* or *down*, + or −; also, when reading on the one side, the divisions on the vernier must be counted in one direction, when reading on the other side, they must be counted in the opposite direction, giving rise to frequent mistakes. It is much better to make the readings always in one way; thus if we place 90° at *horizon, 103° 28′ would* mean 13° 28′ of elevation, 7′

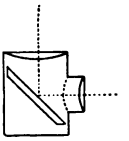
would signify as much of depression. In this way the vernier would mark 0° when the telescope is pointed right downwards or towards the *nadir* (نظير Arabic from نظر to look on); and would mark 180° when the telescope looks right upwards or toward the *zenith*, called in Arabic *semt ul reis*, سمت الرأس (the direction of the head); at 270° the telescope would be again horizontal, 256° 32′ corresponding to an elevation of 13° 28′, while 283° 28′ would correspond to a like depression. If arranged in this way, the instrument would be said to show *nadir distances* or angular distances from the nadir. Astronomers often reckon from the zenith; when the reader is set to 0° at the zenith we have the transition from say 5° on the one side to 355° on the other side; we avoid this inconvenience by placing the zero at the nadir. It is however a matter of taste, and surveyors may prefer to place the zero for the horizontal position of the telescope.

If, after having directed the telescope to some signal and noted the bearing and nadir distance, we turn the horizontal plate G half round, we may bring the telescope again to the same signal. The reading in azimuth should then differ from the former by 180°; also since, in the one position, the nadir distance has been read from right to left, and in the other position from left to right, the sum of the two should be 360°. This affords us the means of ascertaining the index error on the vertical circle, for, since each reading is affected by that error, the sum of the two readings must differ from 360° by its double. It enables us also to test the perpendicularity of the telescope to its axis C D, and the horizontality of that axis without reversing it in its V's.

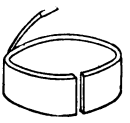
When the signal is near the horizon its apparent bearing *is affected by the error* in the perpendicularity of the tele-

scope to the axis, to the right in one position, to the left in the other, so that the apparent discrepancy between the two azimets is the double of the error. This error may be corrected by the screws of the field-bar. When the signal is at a high elevation, its bearing is also affected by the error in the levelling of the axis C D; to the right in the one position of the instrument, to the left in the other. Instead of attempting to make minute adjustments, it is better to repeat the observations in the two positions of the instrument and to take the average of the results.

In making astronomical observations we have occasion to turn the telescope up to the zenith, and have to use what is called a *diagonal eye-piece*, which is formed by placing a speculum across the eye-piece at an angle of 45°; the eye-lens being placed at the side of the tube.



If, during a dark night, the telescope be turned toward a star, the wires are invisible; in order to make them apparent, a faint light is thrown into the telescope. In properly arranged instruments this is done by means of a small lamp or night-light placed in line at one end of the axis C D. This axis is made hollow to allow the light of the flame to reach a whitened surface placed diagonally across the telescope tube at O, but opened at the middle so as not to intercept the light from the object-glass. The result is a faint haze all over the field of view, against which the wires appear as black lines. We endeavour to do with as little illumination as possible when we are observing small stars, lest these become invisible.



When the instrument has not been furnished with an illuminating apparatus, the surveyor r yet render it available for sidereal observations, by slip

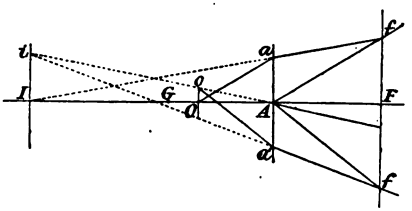
a split-ring on the setting of the object-glass, having previously soldered thereto a bent wire to support a small whitened surface at some little distance in front of the lens. The light of a hand-lamp thrown upon this surface gives the requisite illumination.

LESSON XLVIII.

ON THE COLLIMATING TELESCOPE.

For the adjustment of the theodolite and analogous instruments we have to place a reference signal at a great distance; this is often inconvenient to the surveyor, and almost always so to the instrument maker.

It has been shown in Lesson XXXIII. that if an object O *o* be placed between a lens and its principal focus, G the



light from it, passing through the lens proceeds as if it had come from an image I *i*; the relation of the distances being such that the rectangle under G O and F I is equivalent to the square of A G. Hence if O be placed very close to G, I must be very remote from F; and when O is at G, the image I *i* may be regarded as at an infinite distance, and so, by properly adjusting the distance A O we may cause the object to seem to be at any desired distance; the *arrangement for this* is that already used for the telescope.

Indeed if we adjust a telescope for a very remote object, and then look through its object-glass at the cross-wires, we shall see those as if at a great distance, and we may examine them by help of another telescope; the only condition being that enough of light be admitted at the eye-end to render the wires visible. When two telescopes are so placed, we may look through either at the field-bar of the other; they serve as mutual *collimators*. In this way the telescope of one instrument may serve as collimator for another.

If we place the theodolite between two collimators C D, E F previously put in line with each other, and direct the telescope A B to look into one of them E F, reading the verniers as if making an observation in the field, and then without changing the horizontal circle, turn A B into the opposite direction as to nadir distance, it should collimate C D. If there be an error in azimet the telescope has not been perpendicular to its axis; the two readings in nadir distance should differ by 180°. On turning the instrument half round in azimet, it should again collimate E F at a change of 180° on the horizontal circle; while the reading on the vertical circle should be as much on the one side of 90° or of 270° as the former reading was on the other side. Lastly we may again reverse A B without changing the horizontal position and collimate C D for the second time. These four readings enable us to ascertain the index error in nadir distance.

If the telescope C D be set as for the distinct vision of a *remote object, or as* we say, for parallel rays, and E F be

adjusted to it, E F also is set for parallel rays; and A B when adjusted for distinct vision in the one is also *focussed* for the other. But if C D be lengthened as if to suit a near object, E F must be shortened in order to show the wires distinctly, and A B when adjusted for the one is not adjusted for the other; and thus we have the means of ascertaining whether or not the telescope be set for parallel rays, and for marking the principal focus on the sliding tube of the theodolite telescope, without having had to go out of doors.

LESSON XLIX.

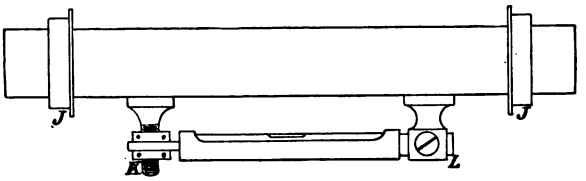
ON THE LEVELLING TELESCOPE.

WHEN the telescope is to be used for levelling we have to place the visual axis horizontally; now in Lesson XLV. we have seen how to make the line of sight correspond with the mechanical axis of the tube, and thus it remains for us to make sure that the mechanical axis is horizontal. For this purpose two distinct methods may be followed; we may attach the spirit-level to the telescope, or we may secure it to the frame which carries the V's.

In the first called the English method two studs are fixed to the telescope tube and are fitted with screws for holding the ends of the level-case, and for adjusting the level to be parallel to the axis. The one of these studs, as at K, has its adjustment to and from the tube, that is vertically, the other, as at L, has its adjustment horizontally in order to place the level and the telescope in one plane.

The adjustment for level is to be tested by reversing the tube end for end in its V's, taking care not to disturb the *position of the stand*; any observed error being corrected

in the usual way by means of the nuts at K. In order to try whether the level be in the same plane with the tube,



we turn the tube slightly round in its V's; if there be any *twist*, as we call it, the air-bubble will move away from the middle, thus indicating the direction of the error, which is to be cured by means of the horizontal screws at L.

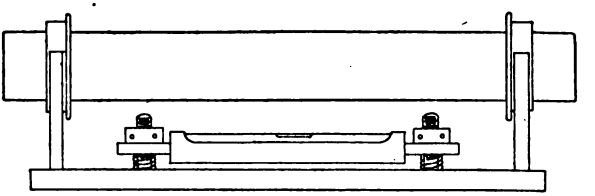
In using this instrument we must carefully observe that the collimation of the wires may not be true for objects at all distances; so that, when great precision is required, as in operations connected with the construction of canals, we must reverse the telescope so as to bring K L, which was below, to be above. Not only so, we should also reverse the telescope end for end in its V's, turn the supporting parts half round, and make a second pair of observations, in order to eliminate all the small errors to which the instrument is liable.

The need for all this care will be understood when we consider an actual case: for example we are to mark the level stakes for a canal of ten, twenty, or thirty miles in length. Here we cannot allow an error of a single foot in the height of the overflows, and therefore we must have a level of sufficient accuracy to show one foot in thirty miles, or one in (roughly) 170 000. If then we take one-tenth part of an inch for the distinctly perceptible motion of the air-bubble, the radius of curvature of its inner surface must be about 17000 inches or 1400 feet.

we be able to observe on the wires to within say the four-thousandth part of an inch, we should require a telescope of 42 inches in focus. For road and railway work such nicety would be quite superfluous.

Now when the telescope is turned so as to have the level-tube above, we have not the means for examining the horizontality at the instant of the observation, and thus half the value of the instrument is lost. The second method, that of securing the spirit-level to the frame, is to be preferred.

In this case the parallelism of the level to the line of vision must be tested by reversing the telescope end for



end, and examining a remote object, or the wires of a collimator, in each position. We may now turn the telescope anyhow on its journals and yet be able to inspect the level. It is advantageous to have a small spirit-level placed crosswise on the frame, whereby to bring the frame horizontal as in the theodolite.

There is an arrangement of the levelling telescope much too commonly in use, in which the telescope tube is not reversible, while the spirit-level is secured to it. Here there is no means within itself of verifying the instrument. If we set up staves at A and B, place the instrument at P equally distant from, but not necessarily between, them; and bringing the air-bubble each time to the middle, make *marks at a and b* where the telescope points on the staves;

a and b are at the same level whether the instrument be in adjustment or not, because there is the same inclination in both cases. If we now remove the instrument to some place Q nearer to B than to A, the effects resulting from any inclination will differ, so that the marks a, β made from the second station Q may not be equidistant from a and b.

But there enter here two new considerations;—firstly, on changing the view from a to β we necessarily elongate the telescope tube; it is rarely indeed that such a motion coincides in direction with the line of vision, so that a new source of error is introduced; secondly, the direction of gravitation at Q is not parallel to its direction at P; the two directions in fact point toward the centre of the earth or rather to the centre of curvature of the earth's surface, wherefore without a knowledge of the earth's dimensions we are not able to estimate the effect of this convergence. Hence the self-verification of such an instrument is difficult if not impossible. We may, however, verify it by using a collimator, provided the collimator be capable of self-verification; in other words, we must use a properly constructed levelling instrument for testing this imperfectly arranged one.

LONDON: PRINTED BY WM. CLOWES AND SONS, STAMFORD STREET AND CHARING